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The moduli space of genus four even spin curves is rational

Hiromichi Takagi [a,*], Francesco Zucconi [b]

[a] Graduate School of Mathematical Sciences, University of Tokyo, Tokyo, 153-8914, Japan
[b] Dipartimento di Matematica e Informatica, Università degli Studî Udine, Via delle Scienze, 206, 33100 Udine, Italy

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Abstract

Using the Mori theory for threefolds, we prove that the moduli space of pairs of smooth curves of genus four and theta characteristics without global sections is a rational variety.

MSC: primary 14H10; 14E30; secondary 14J45; 14N05; 14M20

Keywords: Theta characteristic; Even spin curve; Mori theory; Del Pezzo threefold

1. Introduction

Throughout the paper, we work over $\mathbb{C}$, the complex number field.

The idea to use even spin curves for studies of threefolds or higher dimensional varieties goes back to Tjurin [33]. Mukai was the first to extend this idea [21]. One of his results concerns the geometry of lines on a general smooth prime Fano threefold X of genus 12. He showed that the Hilbert scheme of lines on X is a smooth curve $\mathcal{H}_1$ of genus three, there exists a theta characteristic θ on $\mathcal{H}_1$ without global sections, and X is recovered from the even spin curve $(\mathcal{H}_1, \theta)$ as a certain variety of power sums.

* Corresponding author.
E-mail addresses: takagi@ms.u-tokyo.ac.jp (H. Takagi), zucconi@dimi.uniud.it (F. Zucconi).

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In previous studies we interpreted Mukai's work from the standpoint of quintic normal rational curves on the smooth quintic del Pezzo threefold B, and succeeded in generalizing his results by considering smooth rational curves on B of any degree [31,30]. In these studies, we proceeded in the opposite direction to Tjurin and Mukai, namely, we gave applications of geometries of threefolds to studies of even spin curves. One of the main results is the existence of Scorza quartics for general even spin curves of arbitrary genus [30, Theorem 1.4.1].

In this paper, we move further in this direction and prove the following result.

Theorem 1.1. *The moduli space of even spin curves of genus four is rational.*

In proving Theorem 1.1, the interplay of sextic normal rational curves on B, even spin curves of genus four, and sets of six points on the projective plane modulo PGL_2 action is important. An interesting feature of this interplay is the correspondence of the following:

- a birational selfmap $B \dashrightarrow B$, where the indeterminacy of the map in each direction is a general sextic normal rational curve;
- the interchange of two g^1_3 values of a general curve of genus four;
- the association map between two sets of six points on the projective plane modulo PGL_2 action.

To explain more clearly, we present some notation and conventions. A *spin curve* is a couple of a smooth projective curve of genus g and a theta characteristic θ. There are 2^{2g} different types of spin curve structure for every smooth curve Γ and they are partitioned into two classes according to the parity of $h^0(\Gamma, \theta)$. A theta characteristic θ is said to be *even* or *odd* if $h^0(\Gamma, \theta)$ is even or odd, respectively. Correspondingly we speak of *even or odd spin curves*. The moduli space $\mathcal{S}_g$ of smooth spin curves exists [1,23]. Moreover, studies of the forgetful map $\mathcal{S}_g \to \mathcal{M}_g$, where $\mathcal{M}_g$ is the moduli space of curves of genus g, have revealed that $\mathcal{S}_g$ is a disjoint union of two irreducible components $\mathcal{S}^+_g$ and $\mathcal{S}^-_g$ of relative degrees $2^{g-1}(2^g + 1)$ and $2^{g-1}(2^g - 1)$ corresponding to even and odd spin curves, respectively.

Determination of birational types of $\mathcal{S}^+_g$ is a classic problem. It was classically known that $\mathcal{S}^+_2$ is rational. The so-called Scorza map gives a birational isomorphism between $\mathcal{S}^+_3$ and $\mathcal{M}_3$ [5]; hence, $\mathcal{S}^+_3$ is rational since $\mathcal{M}_3$ is too [2,15]. Farkas [7] and Farkas and Verra [8] proved that a compactification $\overline{\mathcal{S}}^+_g$ of $\mathcal{S}^+_g$ is of general type for $g > 8$, and the Kodaira dimension of $\overline{\mathcal{S}}^+_g$ is negative for $g < 8$ and is zero for $g = 8$. Our result was motivated by these results and the rationality of $\mathcal{M}_4$ [28].

The main technique used to show our result is not invariant theory but threefold Mori theory. In previous studies [31,30] we discovered a method for investigating trigonal even spin curves of any genus using biregular and birational geometries of the smooth quintic del Pezzo threefold B. The threefold B is, by definition, a smooth projective threefold such that $-K_B = 2H$, where H is the ample generator of $\mathrm{Pic}\, B$ and $H^3 = 5$. It is well known that the linear system $|H|$ embeds B into $\mathbb{P}^6$.

First we review our previous results [31,30].

One of our main points is closure of the Hilbert scheme of sextic normal rational curves on B, which we denote by $\mathcal{H}$. We show that $\mathcal{H}$ is an irreducible variety of dimension 12 (Corollary 3.10). We construct a smooth curve $\mathcal{H}_1$ of genus four and a theta characteristic θ on it from a general sextic normal rational curve C on B [31,30]. These arise from the geometry of lines on B intersecting C. It is known that $\mathrm{Aut}\, B$ is isomorphic to the automorphism group PGL_2 of the complex projective line [22,26]. Hereafter, we denote this group by G. The G-action on B induces a G-action on $\mathcal{H}$. Thus, we have a natural rational map $\pi_{\mathcal{S}_4^+}: \mathcal{H} \dashrightarrow \mathcal{S}_4^+$ that maps a general C to $(\mathcal{H}_1, \theta)$ and is constant on general G-orbits. By taking suitable compactifications of $\mathcal{H}$ and of $\mathcal{S}_4^+$, a resolution of indeterminacy of $\pi_{\mathcal{S}_4^+}$, and the Stein factorization, we have rational maps $p_{\mathcal{S}_4^+}: \mathcal{H} \dashrightarrow \widetilde{\mathcal{S}}_4^+$ and $q_{\mathcal{S}_4^+}: \widetilde{\mathcal{S}}_4^+ \dashrightarrow \mathcal{S}_4^+$ such that $\pi_{\mathcal{S}_4^+}$ is given by $q_{\mathcal{S}_4^+} \circ p_{\mathcal{S}_4^+}$, a general fiber of $p_{\mathcal{S}_4^+}$ is connected, and $q_{\mathcal{S}_4^+}$ is generically finite. Then the G-orbit of a general point of $\mathcal{H}$ is contained in a fiber of $p_{\mathcal{S}_4^+}$.

We proved that $\widetilde{\mathcal{S}}_4^+$ is birational to $\mathcal{S}_4^+$ or to its double cover, and birationally parameterizes G-orbits in $\mathcal{H}$ [30, Theorem 4.0.2]. The proof required a detailed study of the birational selfmap $B \dashrightarrow B$ centered along a sextic normal rational curve [30, proof of Lemma 4.0.4], but we provide a refinement here (Theorem 4.6). The selfmap $B \dashrightarrow B$ is decomposed as follows:

$$\begin{tikzcd}
 & A \arrow[r, leftrightarrow, dashed] \arrow[dl, "f"'] & A' \arrow[dr, "f'"] & \\
B \arrow[rrr, leftrightarrow, dashed] & & & B,
\end{tikzcd}$$

where $A \dashrightarrow A'$ is a flop, and f and f' are blow-ups along sextic normal rational curves C and $\widehat{C}$ on B, respectively. We show that $\widetilde{\mathcal{S}}_4^+$ is a double cover to $\mathcal{S}_4^+$ and that the rational deck transformation $J': \widetilde{\mathcal{S}}_4^+ \dashrightarrow \widetilde{\mathcal{S}}_4^+$ of the map $q_{\mathcal{S}_4^+}$ is induced by the correspondence between C and $\widehat{C}$ (Corollary 4.16). An interesting remark is that this J' corresponds to switching of the two g_3^1 on a general curve of genus four.

To show the rationality of $\mathcal{S}_4^+$, we find its good birational model. We previously observed that a general sextic normal rational curve on B has exactly six bi-secant lines [31]. Therefore, noting that the Hilbert scheme of lines on B is $\mathbb{P}^2$, we can define the rational map $\mathcal{H} \dashrightarrow (\mathbb{P}^2)^6/\mathfrak{S}_6$ mapping a general sextic normal rational curve to the unordered set of six points of $\mathbb{P}^2$ corresponding to its six bi-secant lines, where $(\mathbb{P}^2)^6$ is the Cartesian product of six copies of $\mathbb{P}^2$ and $\mathfrak{S}_6$ is the permutation group of its factors. The crucial assertion is that this rational map is birational (Theorem 5.1). It is easy to see that the birational map $\mathcal{H} \dashrightarrow (\mathbb{P}^2)^6/\mathfrak{S}_6$ is G-equivariant, and thus we can translate the study of the rational map $p_{\mathcal{S}_4^+}: \mathcal{H} \dashrightarrow \widetilde{\mathcal{S}}_4^+$ to that of the quotient of $(\mathbb{P}^2)^6/\mathfrak{S}_6$ by G. We carefully choose a G-invariant open subset of $(\mathbb{P}^2)^6/\mathfrak{S}_6$ such that its quotient by G exists and an involution J is induced on the quotient from J'. We denote this quotient by $\mathcal{M}$ only in this introduction. The variety $\mathcal{M}/J$ is birational to $\mathcal{S}_4^+$.

We can study $\mathcal{M}/J$ by relating it to the classically well-studied variety $Y := (\mathbb{P}^2)^6//\mathrm{PGL}_3/\mathfrak{S}_6$, which is a compactification of the moduli space of unordered six distinct points on $\mathbb{P}^2$. First, J has a nice interpretation. It is classically known that Y has an involution called the association map. We show that J is nothing but a lifting of j. Second, the G-action on $\mathbb{P}^2$ realizes G as a closed subgroup of the automorphism group PGL_3 of $\mathbb{P}^2$. G is the subgroup of PGL_3 consisting of elements that preserve one fixed conic on $\mathbb{P}^2$, and hence PGL_3/G is an open subset of $\mathbb{P}^5$. This implies that $\mathcal{M}/J$ is birationally a $\mathbb{P}^5$-bundle over $(X/\mathfrak{S}_6)/j$. It is classically known by Coble that $(X/\mathfrak{S}_6)/j$ is rational and this result was reproved by Dolgachev [4, Appendix].

Therefore, to obtain the rationality of $\mathcal{M}/J$, we only have to show that $\mathcal{M}/J$ is birationally equivalent to $\mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is a locally free sheaf of rank 6 on $(X/\mathfrak{S}_6)/j$. For this, we look for a sub-$\mathbb{P}^4$-bundle of $\mathcal{M} \dashrightarrow X/\mathfrak{S}_6$, which is invariant by J. Then this descends to a sub-$\mathbb{P}^4$-bundle of $\mathcal{M}/J \dashrightarrow (X/\mathfrak{S}_6)/j$ and the local triviality of $\mathcal{M}/J \dashrightarrow (X/\mathfrak{S}_6)/j$ follows. This implies the rationality of $\mathcal{M}/J$ and therefore the rationality of $\mathcal{S}_4^+$. To find the sub-$\mathbb{P}^4$-bundle, we identify the corresponding divisor on $\mathcal{H}$, which is defined by the classes of sextic normal rational curves such that two of their six bi-secant lines intersect.

The remainder of the paper is organized as follows. In Section 2 we review standard results on the smooth quintic del Pezzo threefold B. In particular, we review the behavior of lines on B in detail. In Section 3 we review and supplement our previous results [31,30] for several properties of a general sextic normal rational curve C and the spin curve $(\mathcal{H}_1, \theta)$ associated with C. In Section 4 we establish the correspondence between birational selfmaps whose centers are general sextic normal rational curves and the rational involution on $\widetilde{\mathcal{S}}_4^+$ mentioned above. We construct a birational model of $\mathcal{S}_4^+$ in Section 5 and prove its rationality in Section 6.

2. Auxiliary results for the quintic del Pezzo threefold

In this section, we review results for the quintic del Pezzo threefold. In particular, we review the behavior of lines on it in detail.

2.1. Quintic del Pezzo threefold B

Let $B \subset \mathbb{P}^6$ be the smooth quintic del Pezzo threefold. It is known that B is unique up to projective equivalence and is realized as a linear section of $\mathbb{G}(1,4)$, where $\mathbb{G}(a,b)$ denotes the Grassmannian parameterizing a-dimensional linear varieties of $\mathbb{P}^b$. There are several other characterizations of B. Here we describe one that is suitable for our purpose.

Let $\{\check{F}_2 = 0\} \subset \check{\mathbb{P}}^2$ be a smooth conic. Set

$$\mathrm{VSP}\,(\check{F}_2, 3)^\circ := \{([H_1], [H_2], [H_3]) \mid H_1^2 + H_2^2 + H_3^2 = \check{F}_2\} \subset \mathrm{Hilb}^3\mathbb{P}^2,$$

where $\mathbb{P}^2$ is the dual plane to $\check{\mathbb{P}}^2$, and thus linear forms H_i $(i = 1,2,3)$ can be considered as points in $\mathbb{P}^2$. Mukai showed that B is isomorphic to the closed subset $\mathrm{VSP}\,(\check{F}_2, 3) := \overline{\mathrm{VSP}\,(\check{F}_2, 3)^\circ} \subset \mathrm{Hilb}^3\mathbb{P}^2$ and $\mathbb{P}^2$ is isomorphic to the Hilbert scheme of lines on B [21], [3, Section 4.2]. The variety $\mathrm{VSP}\,(\check{F}_2, 3)$ has the natural action of the subgroup G of the automorphism group PGL_3 of $\mathbb{P}^2$ consisting of elements that preserve $\{\check{F}_2 = 0\}$. The group G is isomorphic to PGL_2, and the conic is the unique one invariant by the action of G. By definition of $\mathrm{VSP}\,(\check{F}_2, 3)^\circ$, it is easy to see that G acts on $\mathrm{VSP}\,(\check{F}_2, 3)^\circ$ transitively. Thus, B is a quasi-homogeneous G-variety. In fact, the automorphism group of B is isomorphic to G [26].

2.2. Lines on B

We summarize results known for lines on B [12,14].

As noted above, the plane $\mathbb{P}^2$ is identified with the Hilbert scheme of lines on B. Moreover, Mukai showed that for a point $b := ([H_1], [H_2], [H_3]) \in \mathrm{VSP}\,(\check{F}_2, 3)^\circ \subset B$, the points $[H_i] \in \mathbb{P}^2$ $(i = 1,2,3)$ represent three lines through b. By the definition of $\mathrm{VSP}\,(\check{F}_2, 3)^\circ$ and the transitivity of the action of G on $\mathrm{VSP}\,(\check{F}_2, 3)^\circ$, it is easy to show the following claim, which is needed in Section 6.

Claim 2.1. *G acts transitively on the set of unordered pairs of intersecting lines whose intersection points are contained in* $\mathrm{VSP}\,(\check{F}_2, 3)^\circ$.

Let F_2 be the quadratic form dual to $\check{F}_2$ and let

$$\Omega := \{F_2 = 0\}$$

be the associated conic in $\mathbb{P}^2$. The conic $\Omega \subset \mathbb{P}^2$ is the unique one invariant under the induced action of G on the Hilbert scheme of lines on B. Moreover, G is exactly the closed subgroup of $\operatorname{Aut}\mathbb{P}^2 \simeq \mathrm{PGL}_3$ whose elements preserve Ω.

There exists a conic Ω' in $\mathbb{P}^2$ such that, for $[l] \in \mathbb{P}^2 - \Omega'$ (resp. $[l] \in \Omega'$), $\mathcal{N}_{l/B} = \mathcal{O}_l \oplus \mathcal{O}_l$ (resp. $\mathcal{N}_{l/B} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$) hold [14, Section 2.5]. Obviously Ω' is invariant under the action of G, and hence we have $\Omega' = \Omega$. Lines parameterized by Ω are called *special lines*. The following assertion was proved by Dolgachev [3, 4.2].

Proposition 2.2. *Let $\widetilde{\Omega}$ be the symmetric bi-linear form associated with Ω. Then two lines l and m on B intersect if and only if $\widetilde{\Omega}([l],[m]) = 0$.*

If this condition holds, then we say that $[l]$ and $[m]$ are polar *with respect to $\widetilde{\Omega}$.*

We need the following claim in Section 6. The proof is easy, so it is omitted.

Claim 2.3. PGL_3/G *is isomorphic to the open subset of $\mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))) \simeq \mathbb{P}^5$ consisting of smooth conics. If we take coordinates x, y, z of $\mathbb{P}^2$ such that $\Omega = \{x^2 + y^2 + z^2 = 0\}$, then the map* $\mathrm{PGL}_3 \to \mathbb{P}^5$ *is induced by $g \in \mathrm{PGL}_3 \mapsto {}^t gg \in \mathbb{P}^5$, where we identify the vector space of symmetric matrices with the vector space of conics on $\mathbb{P}^2$.*

Now we review the description of the universal family of lines and its relations with B. Let

$$\pi : \mathbb{P} \to \mathbb{P}^2$$

be the universal family of lines on B. Explicitly,

$$\mathbb{P} = \{(t, [l]) \mid [l] \in \mathbb{P}^2, t \in l\} \subset B \times \mathbb{P}^2. \tag{2.1}$$

We denote by

$$\varphi : \mathbb{P} \to B$$

the natural projection. As mentioned above, φ is a finite morphism of degree three [12, Lemma 2.3(1)].

Notation 2.4. For an irreducible curve γ on B, let $M(\gamma)$ denote the locus $\subset \mathbb{P}^2$ of lines intersecting γ, namely, $M(\gamma) := \pi(\varphi^{-1}(\gamma))$ with reduced structure. Since φ is flat, $\varphi^{-1}(\gamma)$ is purely one-dimensional. If $\deg \gamma \geq 2$, then $\varphi^{-1}(\gamma)$ does not contain a fiber of π, and thus $M(\gamma)$ is a curve. See Proposition 2.5 for the description of $M(\gamma)$ if γ is a line.

Proposition 2.5. *The following hold:*

(1) *Let B_φ be the union of special lines. Then B_φ is the branched locus of $\varphi : \mathbb{P} \to B$ and has the following properties:*
  (1-1) *$B_\varphi \in |-K_B|$.*
  (1-2) *$\varphi^* B_\varphi = R_1 + 2R_2$, where $R_1 \simeq R_2 \simeq \mathbb{P}^1 \times \mathbb{P}^1$, and $\varphi : R_1 \to B_\varphi$ and $\varphi : R_2 \to B_\varphi$ are injective.*
  (1-3) *The pull-back of a hyperplane section of B to R_1 is a divisor of type* $(1, 5)$.
(2) *The image of R_2 by $\pi : \mathbb{P} \to \mathbb{P}^2$ is the conic Ω.*
(3) *If l is a special line, then $M(l)$ is the tangent line to Ω at l. If l is not a special line, then $\varphi^{-1}(l)$ is the disjoint union of the fiber of π corresponding to l, and the smooth rational curve dominating a line on $\mathbb{P}^2$. In particular, $M(l)$ is the disjoint union of a line and the point l.*

By abuse of notation, *we denote by $M(l)$ the one-dimensional part of $M(l)$ for any line l. Vice versa, any line in $\mathcal{H}_1^B$ is of the form $M(l)$ for some line l.*

(4) *The locus swept by lines intersecting l is a hyperplane section T_l of B whose singular locus is l. For every point b of $T_l - l$, there exists exactly one line that belongs to $M(l)$ and passes through b.*

Proof. See [12,14]. $\square$

According to the proof of Furushima and Nakayama [12], we see that B is decomposed into three G-orbits as follows:

$$B = (B - B_\varphi) \cup (B_\varphi - C_\varphi) \cup C_\varphi,$$

where C_φ is a sextic normal rational curve: if $b \in B - B_\varphi$, exactly three distinct lines pass through it; if $b \in (B_\varphi - C_\varphi)$, exactly two distinct lines pass through it, one of which is special; and C_φ is the unique closed G-orbit and is the loci of $b \in B$ through which only one line passes, which is special. It holds that $\mathrm{VSP}^\circ(F_2, 3) = B - B_\varphi$.

Finally, we describe the linear projection of B from a line.

Proposition 2.6. *The target of the linear projection of B from a line l is the smooth quadric threefold Q. Moreover, the projection is decomposed as follows:*

$$\begin{tikzcd} & B_l \arrow[dl, "\pi_{1l}"'] \arrow[dr, "\pi_{2l}"] & \\ B \arrow[rr, dashed] & & Q, \end{tikzcd}$$

where π_{1l} is the blow-up along l, $B \dashrightarrow Q$ is the linear projection from l, and π_{2l} contracts onto a twisted cubic curve γ, the strict transform of the locus T_l swept by the lines of B intersecting l. A nontrivial fiber of π_{2l} is the strict transform of a line intersecting l. Moreover, we have the following description:

(1)

$$-K_{B_l} = H_l + L_l,$$

where H_l and L_l are the pull-backs of general hyperplane sections of B and Q, respectively. We denote by E_l the π_{1l}-exceptional divisor.

(2) *A line m on B disjoint from l is mapped to a line m' on Q such that m' intersects γ at one point simply and m' is not contained in the hyperplane section spanned by γ, and vice versa.*

Proof. The proof is well known and is explicitly stated by Fujita. See [9]. $\square$

3. Sextic rational curves on B and even spin curves of genus four

Definition 3.1. We inductively define $\mathcal{H}_d^B$ to be the union of the components of the Hilbert scheme whose general point parameterizes a smooth rational curve of degree d on B obtained as a smoothing of the union of a general smooth rational curve of degree $d-1$ belonging to $\mathcal{H}_{d-1}^B$ and its general uni-secant line.

We previously studied several properties of a general C_d belonging to $\mathcal{H}_d^B$ [31] and constructed the spin curve $(\mathcal{H}_1, \theta)$ associated with C_d [30]. In this section, we review the results and supplement some properties of C_d that are required in this paper. We often denote C_d simply by C.

3.1. Properties of general rational curves on B of degree ≤ 6

Proposition 3.2. *A general element $[C] \in \mathcal{H}_d^B$ for any d satisfies the following conditions:*

(1) *There exist no k-secant lines of C on B with $k \geq 3$.*
(2) *There exist at most finitely many bi-secant lines of C on B, and any of them intersects C simply.*
(3) *No bi-secant lines of C are special lines.*
(4) *Bi-secant lines of C are mutually disjoint.*

Proof. This has already been proved [31, Proposition 2.3.1]. In fact, condition (1) holds if C is a normal rational curve since it is the intersection of quadrics containing it. $\square$

We review some additional relations of C with lines on B that can be translated into the geometry of $M(C)$ in $\mathbb{P}^2$ (cf. Notation 2.4). We denote by β_i $(1 \leq i \leq s)$ the bi-secant lines of a general C belonging to $\mathcal{H}_d^B$.

Proposition 3.3. *A general element $[C] \in \mathcal{H}_d^B$ for any d satisfies the following conditions:*

(1) *C intersects B_φ simply.*
(2) *$M(C)$ is an irreducible curve of degree d with only simple nodes, and all the nodes correspond to bi-secant lines of C (if $d = 1$, then recall that in Proposition 2.5(3) we abuse the notation by denoting the one-dimensional part of $\pi(\varphi^{-1}(C))$ by $M(C)$).*
(3) *For a general line l intersecting C, $M(C) \cup M(l)$ has only simple nodes as its singularities.*
(4) *$M(C) \cup M(\beta_i)$ has only simple nodes as its singularities.*

Proof. This has already been proved [31, Proposition 2.3.3]. $\square$

Now we focus on the case in which $d = 6$ and supplement some properties of a general $[C] \in \mathcal{H}_6^B$.

By Proposition 3.3(2), $M(C)$ is a nodal plane sextic curve. We can count the number of nodes of $M(C)$.

Corollary 3.4. *The number of nodes of $M(C)$ is six, and thus C has six bi-secant lines on B.*

Proof. This has already been proved [31, Corollary 4.1.2]. $\square$

We denote by

$$\beta_1, \ldots, \beta_6$$

the six bi-secant lines of C.

Corollary 3.5. *For a general $[C] \in \mathcal{H}_6^B$,*

(1) *there are two lines α_{i1} and α_{i2} intersecting both C and β_i outside $C \cap \beta_i$ $(1 \leq i \leq 6)$;*
(2) *there are two lines γ_{ij1} and γ_{ij2} intersecting both C and α_{ij} outside $C \cap \alpha_{ij}$, and they are disjoint from each other ($i = 1, \ldots, 6$, $j = 1, 2$).*

Proof. We first show assertion (1). By Proposition 3.3(2) and (4), there exist six different lines on B intersecting both C and β_i for any fixed $i = 1, \ldots, 6$, and they are not bi-secant lines. Since β_i intersects C at two points, four of the six lines pass through $\beta_i \cap C$. The remaining two lines are exactly the desired lines α_{i1} and α_{i2}.

For assertion (2), the two lines γ_{ij1} and γ_{ij2} are obtained in a manner similar to the proof of (1). Proof of the property $\gamma_{ij1} \cap \gamma_{ij2} = \emptyset$ is given by a simple dimension count as in [31, proof of Propostion 2.3.1], so it is omitted. $\square$

The following generality of six points corresponding to six bi-secant lines links to the classical results of algebraic geometry in Section 5.

Proposition 3.6. *The six points $[\beta_1], \dots, [\beta_6]$ on $\mathbb{P}^2$ are in a general position.*

Proof. We show there is no line in $\mathbb{P}^2$ through three of $[\beta_1], \dots, [\beta_6]$. Assume by contradiction that there exists a line L through the three points $[\beta_{i_1}], [\beta_{i_2}]$ and $[\beta_{i_3}]$. By Proposition 2.5(3), there exists a line l on B such that $M(l) = L$. The above condition means that the three bi-secant lines β_{i_1}, β_{i_2} and β_{i_3} intersect l.

Consider the linear projection $B \dashrightarrow Q$ from β_{i_1} as in Proposition 2.6 and let C', β'_{i_2}, and β'_{i_3} be the images on Q of C, β_{i_2}, and β_{i_3}, respectively. The degree of C' is four since β_{i_1} is a bi-secant line of C. The curves β'_{i_2} and β'_{i_3} are lines since β_{i_2} and β_{i_3} are disjoint from β_{i_1}. Moreover, since $\beta_{i_1} \cap l \neq \emptyset$, the line l is mapped to a point $p \in Q$, and since $\beta_{i_2} \cap l \neq \emptyset$ and $\beta_{i_3} \cap l \neq \emptyset$, the lines β'_{i_2} and β'_{i_3} intersect at p. The lines β'_{i_2} and β'_{i_3} are bi-secant lines of C'.

Now we consider the linear projection $Q \dashrightarrow \mathbb{P}^2$ from β'_{i_2} and let C'' be the image of C' on the target $\mathbb{P}^2$. Since β'_{i_2} is a bi-secant line of C', C'' is a line and $C' \to C''$ is of degree two, or C'' is a conic and $C' \to C''$ is an isomorphism. Since $\beta'_{i_2} \cap \beta'_{i_3} \neq \emptyset$ and β'_{i_3} is a bi-secant line of C', the morphism $C' \to C''$ cannot be an isomorphism. Therefore, C'' is a line. Then, however, C is contained in the hyperplane section of B, which is mapped to C'', a contradiction.

We previously showed that there are no conics through the six points $[\beta_1], \dots, [\beta_6]$ using the inductive construction of C [30, proof of Lemma 3.1.1]. $\square$

3.2. Irreducibility of $\mathcal{H}^B_d$

The contents in this subsection are new. We investigate the irreducibility of the Hilbert schemes of rational curves on B with some additional conditions.

Proposition 3.7. *For any d, $\mathcal{H}^B_d$ is irreducible and is of dimension $2d$.*

Proof. The claim is true for $d = 1$, since $\mathcal{H}^B_1 \simeq \mathbb{P}^2$. By induction, assume that $\mathcal{H}^B_{d-1}$ is irreducible. Let $[C_{d-1}] \in \mathcal{H}^B_{d-1}$ be a generic element. The family of lines $[l] \in \mathbb{P}^2$ that intersect a general element of $\mathcal{H}^B_{d-1}$ is irreducible by Proposition 3.3(2). This implies that the family $\mathcal{H}^B_{d-1,1}$ of reducible curves $C^0_d = C_{d-1} \cup l$ such that $[C_{d-1}] \in \mathcal{H}^B_{d-1}$, $[l] \in \mathbb{P}^2$ and length $C^0_{d-1} \cap l = 1$ is irreducible. The Hilbert scheme is smooth at point C^0_d [31, proof of Proposition 2.2.2]. Therefore, $\mathcal{H}^B_d$ is irreducible.

The assertion that $\dim \mathcal{H}^B_d = 2d$ follows from our previous study [31, Proposition 2.2.2]. $\square$

We refine this assertion for $d \leq 6$.

For a smooth projective variety X in some projective space, let $\mathcal{H}^0_d(X)$ be the union of components of the Hilbert scheme whose general points parameterize smooth rational curves on X of degree d. By [25], $\mathcal{H}^0_d(\mathbb{G}(a,b))$ is non-empty and irreducible for the Grassmann variety $\mathbb{G}(a,b)$.

Let $\mathcal{H}^{0'}_d(X)$ be the open subset of $\mathcal{H}^0_d(X)$ parameterizing smooth rational curves on X of degree d with linear hulls of maximal dimension.

We can show inductively that $\mathcal{H}^B_d \subset \overline{\mathcal{H}^{0'}_d(B)}$, where we take the closure in the Hilbert scheme. Therefore, we can ask the following.

Question 3.8. $\mathcal{H}_d^B = \overline{\mathcal{H}_d^{0'}(B)}$? Are they irreducible?

We have a partial answer to this question as follows.

Proposition 3.9. $\mathcal{H}_d^B = \overline{\mathcal{H}_d^{0'}(B)}$ *for* $d \le 6$.

Proof. We only have to show that $\mathcal{H}_d^{0'}(B)$ with $d \le 6$ is irreducible. Note that for $d \le 6$, $\mathcal{H}_d^{0'}(B)$ is nothing but the Hilbert scheme of normal rational curves of degree d [31, Corollary 2.2.3]. Let U be the open subset of $\mathbb{G}(\mathbb{P}^6, \mathbb{P}^9)$ consisting of points $[P] \in \mathbb{G}(\mathbb{P}^6, \mathbb{P}^9)$ such that the six-plane P is transversal to $\mathbb{G}(1,4)$, and let $\mathcal{P} \to U$ be the universal family over U of $\mathbb{P}^6$'s in $\mathbb{P}^9$. Let $\mathcal{B} := \mathcal{P} \cap (\mathbb{G}(1,4) \times U)$, which is an irreducible family of smooth quintic del Pezzo threefolds.

Consider the incidence variety

$$\mathcal{J} := \{(C_d^0, B) \in \mathcal{H}_d^{0'}(\mathbb{G}(1,4)) \times \mathcal{B} \mid C_d^0 \subset B\}.$$

A general fiber $\mathcal{J} \to \mathcal{B}$ is equal to $\mathcal{H}_d^{0'}(B)$. Moreover, any fiber of $\mathcal{J} \to \mathcal{H}_d^{0'}(\mathbb{G}(1,4))$ is isomorphic to $\mathbb{G}(\mathbb{P}^d, \mathbb{P}^6)$. Since $\mathcal{H}_d^{0}(\mathbb{G}(1,4))$ is irreducible and $\mathcal{H}_d^{0'}(\mathbb{G}(1,4))$ is an open subset of $\mathcal{H}_d^{0}(\mathbb{G}(1,4))$, it holds that $\mathcal{J}$ is irreducible. According to a previous argument [19, proof of Theorem 3.1], we only have to show that there is one particular component of a general fiber $\mathcal{J} \to \mathcal{B}$ invariant under monodromy. In fact, this is nothing but $\mathcal{H}_d^B$. □

As a summary of the results for the case in which $d = 6$, we state the following.

Corollary 3.10. $\mathcal{H}_6^B$ *is the closure of the Hilbert scheme of sextic normal rational curves on* B*, and is an irreducible variety of dimension* 12*.*

From now on we set

$$\mathcal{H} := \mathcal{H}_6^B.$$

3.3. *Even spin curve* $(\mathcal{H}_1, \theta)$ *of genus four*

Although the argument in this subsection also applies to other degrees d, we focus on the degree six case.

Throughout Section 3.3, we assume that $[C]$ is a general point of $\mathcal{H} := \mathcal{H}_6^B$.

3.3.1. *Construction of* $\mathcal{H}_1$

We set

$$\mathcal{H}_1 := \varphi^{-1}C \subset \mathbb{P}. \tag{3.1}$$

By definition, $\mathcal{H}_1$ parameterizes pairs of lines l on B and points $t \in C \cap l$ (cf. (2.1)). We call such a pair (l, t) a *marked line* on B [31].

Proposition 3.11. $\mathcal{H}_1$ *is a smooth non-hyperelliptic trigonal curve of genus four.*

Proof. We have already proved this [31, Corollary 4.1.1]. We only note that the morphism $\varphi_{|\mathcal{H}_1} : \mathcal{H}_1 \to C$ corresponds to one of g_3^1 since three lines pass through one point of B and $C \simeq \mathbb{P}^1$. □

3.3.2. Concept of a line on the blow-up A of B along C

Let

$$f: A \to B$$

be the blow-up of B along C. We introduce the concept of a line on A, which is a special type of curve of anticanonical degree one. This is useful for another interpretation of the curve $\mathcal{H}_1$.

We denote by

$$E \subset A$$

the f-exceptional divisor and by

$$\beta'_1, \dots, \beta'_6$$

the strict transforms on A of the bi-secant lines of C.

Notation 3.12. For $i = 1, \dots, s$ and $j = 1, 2$, we set

(1) $\{p_{i1}, p_{i2}\} = C \cap \beta_i \subset B$, and
(2) $\zeta_{ij} = f^{-1}(p_{ij}) \subset E \subset A$.

Definition 3.13. We say that a connected curve $l \subset A$ is a *line* on A if $-K_A \cdot l = 1$ and $E \cdot l = 1$.

Remark. Since $\rho(A) = 2$, the numerical class of lines on A is unique.

We point out that since $-K_A = f^*(-K_B) - E$ and $E \cdot l = 1$, then $f(l)$ is a line on B *intersecting* C. On this basis, we can classify lines on A as follows.

Proposition 3.14. *A line l on A is one of the following curves on A:*

(i) *the strict transform of a uni-secant line of C on B, or*
(ii) *the union $l_{ij} := \beta'_i \cup \zeta_{ij}$ ($i = 1, \dots, s$, $j = 1, 2$).*

In particular, l is reduced and $p_a(l) = 0$.

Proposition 3.15. *The curve $\mathcal{H}_1 \subset \mathbb{P}$ is isomorphic to the Hilbert scheme of A parameterizing lines on A.*

Proof. We only show that $\mathcal{H}_1$ parameterizes lines on A. See [31, Corollary 4.1.8] for a rigorous proof.

According to the definition of $\mathcal{H}_1$ in (3.1), we note that $\mathcal{H}_1$ parameterizes marked lines. It is easy to see that there is one-to-one correspondence between marked lines and lines on A. Indeed, let m be a line on A. The line m satisfies (1) or (2) of Proposition 3.14. If m satisfies (1), then the image $f(m)$ of m on B is a uni-secant line, and thus a marked line $(f(m), C \cap m)$ is uniquely determined from m. If $m = \beta'_i \cup \zeta_{ij}$, then we assign the marked line (β_i, p_{i3-j}) to m (Notation 3.12). Therefore $\mathcal{H}_1$ parameterizes lines on A. □

By the proof of Proposition 3.15 and [31, Remark after Claim 4.1.9], we have the following.

Corollary 3.16. $\pi^{-1}_{|\mathcal{H}_1}([\beta_i]) = \{[l_{i1}], [l_{i2}]\}$ *for $i = 1, \dots, 6$. Moreover, the point $([\beta_i], p_{ij})$ of $\mathcal{H}_1$ corresponds to $[l_{i\,3-j}]$ ($1 \le i \le 6$, $j = 1, 2$) (Notation 3.12 and Proposition 3.14).*

We use the following two results on $\beta'_1, \dots, \beta'_6$.

Proposition 3.17. *$\beta_1', \dots, \beta_6'$ are disjoint.*

Proof. We show the following more general result. Let $C \subset \mathbb{P}^d$ be a normal rational curve of degree d and let $\widetilde{\mathbb{P}}^d \to \mathbb{P}^d$ be the blow-up of $\mathbb{P}^d$ along C. Then the strict transforms of bi-secant lines on $\widetilde{\mathbb{P}}^d$ are disjoint. This assertion follows from a general result for the secant scroll of a normal rational curve in the ambient projective space [34, Theorem 3.9]. In fact, the blow-up of the secant scroll along the normal rational curve is a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$ and its fibers are the strict transforms of bi-secant lines. In particular, they are disjoint. □

Proposition 3.18. *It holds that $\mathcal{N}_{\beta_i'/A} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$.*

Proof. We have already proved this [31, Lemma 5.1.6]. □

3.3.3. Theta characteristic on $\mathcal{H}_1$

Via the new interpretation of $\mathcal{H}_1$ (Proposition 3.15), we define the following incidence correspondence:

$$I := \{([l_1], [l_2]) \in \mathcal{H}_1 \times \mathcal{H}_1 \mid l_1 \neq l_2 \text{ and } l_1 \cap l_2 \neq \emptyset\}.$$

In fact, we have already provided a scheme theoretic definition [30, Section 3.1]. We denote by δ the g_3^1 on $\mathcal{H}_1$ that defines $\varphi_{|\mathcal{H}_1}: \mathcal{H}_1 \to C$. Let t be a general point of C and let l, l' and l'' be three lines on A such that $\varphi(l), \varphi(l')$ and $\varphi(l'')$ are three lines passing through t. By definition, we have $[l] + [l'] + [l''] = \varphi^{-1}(t) \sim \delta$. Set

$$\theta := (\pi_{|\mathcal{H}_1})^* \mathcal{O}_{M(C)}(1) - \delta. \tag{3.2}$$

Note that $\deg \theta = \deg M(C) - 3 = 3$.

Proposition 3.19. *The class of θ is an ineffective theta characteristic and $I = I_\theta$, where, by definition, $(x, y) \in I_\theta$ if and only if y belongs to the support of the unique effective divisor in $|\theta + x|$.*

Proof. We have already proved this [30, Proposition 3.1.2]. □

In summary, we have constructed a rational map $\mathcal{H} \dashrightarrow \mathcal{S}_4^+$ associating a general $[C] \in \mathcal{H}$ with $[(\mathcal{H}_1, \theta)] \in \mathcal{S}_4^+$.

3.4. Good open subset $\mathcal{H}^\circ$ of $\mathcal{H}$

As a summary of several properties of general smooth sextic rational curves, we define the following open subset $\mathcal{H}^\circ$ of $\mathcal{H}$.

Condition 3.20. Let $\mathcal{H}^\circ$ be the open set of $\mathcal{H}$ consisting of smooth sextic rational curves C that satisfy all the following conditions of generality. We can check that each condition is satisfied for a general C, and thus $\mathcal{H}^\circ \neq \emptyset$. We indicate below the place where each condition is checked for a general C:

(a) C is a normal rational curve, namely, C spans $\mathbb{P}^6$ [31, Corollary 2.2.3].
(b) C has exactly six different bi-secant lines $\beta_1, \dots, \beta_6$ (Corollary 3.4).
(c) $M(C)$ is an irreducible plane sextic curve, and $\operatorname{Sing} M(C)$ consists of six nodes $[\beta_1], \dots, [\beta_6]$ (Proposition 3.3(2) and Corollary 3.4).

Note that this condition and condition (b) imply that $\mathcal{H}_1$ is smooth and C does not have a tangent line in B.

(d) $[\beta_1], \dots, [\beta_6] \in \mathbb{P}^2$ are in a general position (Proposition 3.6).

(e) $\theta := (\pi_{|\mathcal{H}_1})^* \mathcal{O}_{M(C)}(1) - \delta$ is a theta characteristic, and $h^0(\mathcal{H}_1, \theta) = 0$ (Proposition 3.19).

(f) For the strict transform β_i' on A of β_i ($i = 1, \dots, 6$), it holds that $\mathcal{N}_{\beta_i'/A} = \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$ (Proposition 3.18).

(g) Six bi-secant lines are mutually disjoint (Proposition 3.2(4)).

(h) For $i = 1, \dots, 6$, there are two lines α_{i1} and α_{i2} intersecting both C and β_i outside $C \cap \beta_i$ (Corollary 3.5(1)).

(i) Any point in $\beta_i \cap C$ is not contained in B_φ. More explicitly, for $i = 1, \dots, 6$, there are two lines different from β_i through each of the intersection point of C and β_i.
Note that this condition implies that no bi-secant lines of C are special lines.
To check this condition for a general C, we need Theorem 4.6 and Lemma 4.8, which we show below. Indeed, take $\widehat{C}$ as in Theorem 4.6 for a general C. Then $\widehat{C}$ satisfies (i) by condition (j) for C and Lemma 4.8.

(j) For $i = 1, \dots, 6$ and $j = 1, 2$, there are two lines γ_{ij1} and γ_{ij2} different from β_i such that they intersect both C and α_{ij} and their strict transforms on A intersect the strict transform of α_{ij} (Corollary 3.5(2)).
Note that by conditions (h) and (i), none of γ_{ij1} and γ_{ij2} intersects β_i.

Note that the rational map $\mathcal{H} \dashrightarrow \mathcal{S}_4^+$ is a morphism on $\mathcal{H}^\circ$.

Remark. Conditions 3.20(a)–(h) are more or less essential conditions for our method. Conditions (i) and (j) are slightly technical; condition (j) for C is needed to verify (h) for $\widehat{C}$ as in Theorem 4.6, and condition (i) for C is needed to verify (j) for $\widehat{C}$.

4. Birational selfmap of B and rational involution on $\widetilde{\mathcal{S}}_4^+$

In this section, we establish the correspondence between birational selfmaps whose centers are general sextic normal rational curves and the rational involution on $\widetilde{\mathcal{S}}_4^+$, as mentioned in the Introduction.

4.1. Smooth threefold flops

For convenience, we provide the definition and basic properties of flops.

Definition 4.1. Let A be a smooth threefold. A projective morphism $g: A \to \overline{A}$ is called a *flopping contraction* if g is isomorphic outside the union γ of a finite number of curves (actually each connected component of γ is a tree of smooth rational curves) and any irreducible component of γ is numerically trivial for K_A. An irreducible component of γ is called a *flopping curve*. If there exists a divisor D numerically negative for any irreducible component of γ, then g is called a *D-flopping contraction*. It is well known that for a D-flopping contraction g, there exists a unique projective morphism $g': A' \to \overline{A}$ such that [17]

- g' is an isomorphism outside the union γ' of a finite number of curves and any irreducible component of γ' is numerically trivial for $K_{A'}$.
- The map $g'^{-1} \circ g: A \dashrightarrow A'$ gives an isomorphism between $A - \gamma$ and $A' - \gamma'$.
- The strict transform D' on A' of D is numerically *positive* for any irreducible component of γ'.

The map $g'^{-1} \circ g: A \dashrightarrow A'$ is called the *D-flop* for g and the morphism g' is called the *D-flopped contraction*. An irreducible component of γ' is called a *flopped curve*.

If $\rho(A/\overline{A}) = 1$ (e.g., γ is irreducible), then the D-flop is independent of D and we say simply $A \dashrightarrow A'$ is the flop, g' is the flopped contraction, etc.

In Proposition 4.2, we summarize basic properties of flops, for which it is easy to find references in the literature.

Proposition 4.2. *Let A be a smooth threefold and D a divisor on A. Let $g: A \to \overline{A}$ be a D-flopping contraction and γ the union of all the flopping curves. Let $A \dashrightarrow A'$ be the D-flop and $g: A' \to \overline{A}$ the D-flopped contraction. We denote by γ' the union of all the D-flopped curves. Then*

(1) *A' is smooth.*
(2) *g and g' are isomorphic to each other analytically near γ and γ'. In particular, the numbers of irreducible components of γ and γ' are equal.*
(3) *If $\rho(A/\overline{A}) = 1$, then $G \cdot \gamma = -G' \cdot \gamma'$, where G is a divisor on A and G' is the strict transform on A' of G.*

Proof. See [17]. $\square$

Example 4.3 (*Atiyah's Flop*)**.** We now describe the simplest flopping contraction. In what follows we mainly need only (composites of) flopping contractions of this type.

Let $g: A \to \overline{A}$ be a projective morphism whose exceptional curve γ is a smooth irreducible rational curve with $\mathcal{N}_{\gamma/A} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$. It is easy to check that g is a flopping contraction. We can construct the flop $A \dashrightarrow A'$ as follows. Let $p: \widehat{A} \to A$ be the blow-up of A along γ and let E be the p-exceptional divisor. Since $\mathcal{N}_{\gamma/A} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$, it holds that $E \simeq \mathbb{P}^1 \times \mathbb{P}^1$. There exists a morphism $q: \widehat{A} \to A'$ that is isomorphic outside E and $q_{|E}$ is the natural projection $E \simeq \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ different from $E \to \gamma$. It is easy to check that there exists a projective morphism $g': A' \to \overline{A}$ that is isomorphic outside $\gamma' := q(E)$ and $q \circ p^{-1}: A \dashrightarrow A'$ is the flop. The flop $A \dashrightarrow A'$ is called *Atiyah's flop*.

$$\begin{tikzcd}
 & \widehat{A} \arrow[dl, "p"'] \arrow[dr, "q"] & \\
A \arrow[rr, dashed, leftrightarrow] \arrow[dr, "g"'] & & A'. \arrow[dl, "g'"] \\
 & \overline{A} &
\end{tikzcd} \tag{4.1}$$

By abuse of convention, a flop with several disjoint exceptional curves with normal bundles of type $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$ is also called Atiyah's flop.

The following two results, Propositions 4.4 and 4.5, describe changes in intersection numbers by a flop.

Proposition 4.4. *Let A be a smooth threefold and let $g: A \to \overline{A}$ be a flopping contraction with $\rho(A/\overline{A}) = 1$. We denote by γ the union of all the g-exceptional curves. On A, take a divisor N and an irreducible projective curve $\delta \not\subset \gamma$. Let $A \dashrightarrow A'$ be the flop, and let N' and δ' be the strict transforms on A' of N and δ respectively. The following hold.*

(1) *If $N \cdot \gamma = 0$, then $N^3 = {N'}^3$ and $N \cdot \delta = N' \cdot \delta'$.*
(2) *If $N \cdot \gamma > 0$* (resp. $N \cdot \gamma < 0$)*, then $N^3 > {N'}^3$ and $N \cdot \delta \leq N' \cdot \delta'$* (resp. *$N^3 < {N'}^3$ and $N \cdot \delta \geq N' \cdot \delta'$*).

Proof. By Proposition 4.2, the inverse $A' \dashrightarrow A$ of $A \dashrightarrow A'$ is also the flop for the flopping contraction $g' : A' \to \overline{A}$, and thus we can assume that $N \cdot \gamma \geq 0$ by interchanging the roles of A and A'. Then the inequality between N^3 and ${N'}^3$ follows from [29, Corollary 9.3], and the inequality between $N \cdot \delta$ and $N' \cdot \delta'$ follows from the so-called negativity lemma [18, Lemma 2.19]. □

We frequently use the following refinement of Proposition 4.4 for Atiyah's flops.

Proposition 4.5. *Let A be a smooth threefold and let $g : A \to \overline{A}$ be a flopping contraction whose exceptional curve γ is irreducible. Assume that $\mathcal{N}_{\gamma/A} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Let N be a divisor on A and set $d := N \cdot \gamma$. Let δ be a smooth irreducible projective curve different from γ and let e be the set-theoretic intersection number of δ and γ. Let $A \dashrightarrow A'$ be the flop, and let N' and δ' be the strict transforms on A' of N and δ, respectively.*

It holds that $N^3 = (N')^3 + d^3$ and $N' \cdot \delta' \geq N \cdot \delta + de$. Moreover, if γ and δ intersect transversely at e points, then $N' \cdot \delta' = N \cdot \delta + de$.

Proof. Consider diagram (4.1) in Example 4.3. We can write $q^*N' = p^*N + aE$ with some $a \in \mathbb{Z}$. We show that $a = d$. For a fiber $\widehat{\gamma}$ of $E \to \gamma'$, which is mapped to γ by p, it holds that $q^*N' \cdot \widehat{\gamma} = 0$, $p^*N \cdot \widehat{\gamma} = N \cdot \gamma = d$, and $E \cdot \widehat{\gamma} = -1$. Therefore, we have $a = d$.

Now we prove the inequality $N' \cdot \delta' \geq N \cdot \delta + de$. Let $\widehat{\delta}$ be the strict transform on $\widehat{A}$ of δ. By definition of e, it holds that $E \cdot \widehat{\delta} \geq e$. By $q^*N' = p^*N + dE$, we have

$$N' \cdot \delta' = q^*N' \cdot \widehat{\delta} = (p^*N + dE) \cdot \widehat{\delta} \geq p^*N \cdot \widehat{\delta} + de = N \cdot \delta + de.$$

Moreover, if γ and δ intersect transversely at e points, then it holds that $E \cdot \widehat{\delta} = e$. Thus, we have $N' \cdot \delta' = N \cdot \delta + de$.

To prove the equality $N^3 = (N')^3 + d^3$, we compute $p^*N^2 q^*N'$ in two ways. First, by applying the projection formula to p, we have $p^*N^2 q^*N' = N^3$. Second, by the equality $p^*N = q^*N' - dE$, we have

$$p^*N^2 q^*N' = (q^*N' - dE)^2 q^*N' = (q^*N')^3 + d^2E^2 q^*N' = (N')^3 + d^2 N' \cdot q_*(E^2),$$

for which $(q^*N')^2 E = (q^*N'_{|E})^2 = 0$ holds since E is a $\mathbb{P}^1$-bundle over a curve and $q^*N'_{|E}$ is numerically the sum of its fibers. Thus, we have $N^3 = (N')^3 + d^2 N' \cdot q_*(E^2)$. It is easy to see that $-q_*(E^2) = \gamma'$ is a 1-cycle. Therefore, $N' q_*(E^2) = -N' \cdot \gamma' = N \cdot \gamma = d$ by Proposition 4.2(3). Consequently, we have the equality $N^3 = (N')^3 + d^3$. □

4.2. *Birational selfmap of B*

In this subsection, we describe the birational selfmap of B whose center is a general sextic rational curve. This description is inspired by the work of Takeuchi [32]. The self-contained proof of the following result is long and is included in Appendix to avoid breaking the flow. We hope the proof is a good introduction to the explicit threefold Mori theory.

To understand the proof of Theorem 1.1, readers need only understand the following statement.

Theorem 4.6. *Let C be a sextic normal rational curve on B and let $f : A \to B$ be the blow-up along C. We denote by E the f-exceptional divisor. If C has only a finite number of bi-secant lines (Proposition 3.2(2)) and C is not contained in B_φ (Proposition 3.3(1)), then we have the* *following diagram:*

$$\begin{tikzcd} & A \arrow[r, dashed, leftrightarrow] \arrow[dl, "f"'] & A' \arrow[dr, "f'"] & \\ B \arrow[rrr, dashed, leftrightarrow] & & & B, \end{tikzcd} \tag{4.2}$$

where $A \dashrightarrow A'$ is one flop and $f' : A' \to B$ is the blow-up along a sextic normal rational curve $\widehat{C}$. We denote by E' the f'-exceptional divisor. We also denote by H (resp. *L*) *the pull-back by f* (resp. *f'*) *of a general hyperplane section of B on the left-hand (resp. right-hand) side. For simplicity, we denote the strict transforms on A' of curves and divisors on A using the same notation. It holds that*

$$L = 3H - 2E, \qquad -2K_A = H + L \quad \textit{and} \quad E' = 4H - 3E. \tag{4.3}$$

Suppose $[C] \in \mathcal{H}^\circ$ (Condition 3.20). Then C has only a finite number of bi-secant lines and C is not contained in B_φ, and hence the above assertions hold. Moreover, all the flopping curves of $A \dashrightarrow A'$ are the strict transforms $\beta'_1, \dots, \beta'_6$ of six bi-secant lines $\beta_1, \dots, \beta_6$ of C.

Corollary 4.7. *Let l be a line on B intersecting C disjoint from the images of the flopping curves. Then the strict transform of l on B on the right-hand side of (4.2) is a line intersecting $\widehat{C}$.*

Proof. Let l' be the strict transform of l on A. Since $E' = 4H - 3E$, we have $E' \cdot l' = 1$ on A. Since l' is disjoint from the flopping curves, we have $-K_{A'} \cdot l' = 1$ and $E' \cdot l' = 1$ on A', and thus the image of l' by f' is a line intersecting $\widehat{C}$. □

4.3. *Rational involution on $\widetilde{\mathcal{S}}_4^+$*

We presented the content of this subsection previously [30, proof of Lemma 4.0.5]; here we need to refine the proof for later use.

Throughout Section 4.3, we assume $[C] \in \mathcal{H}^\circ$ (Condition 3.20). By Theorem 4.6, we have another sextic normal rational curve $\widehat{C}$ on B. The goal of this subsection is to show $[\widehat{C}] \in \mathcal{H}^\circ$ (Claims 4.9, 4.10, 4.13 and 4.15). Then we see that the correspondence $[C] \mapsto [\widehat{C}]$ defines an involution on the image of $\mathcal{H}^\circ$ in $\widetilde{\mathcal{S}}_4^+$ (Corollary 4.16).

For simplicity, we denote the strict transforms on A' of curves and divisors on A using the same notation.

We denote by

$$\beta''_1, \dots, \beta''_6$$

the flopped curve on A' corresponding to $\beta'_1, \dots, \beta'_6$, and by

$$\widehat{\beta}_1, \dots, \widehat{\beta}_6$$

the images of the flopped curves $\beta''_1, \dots, \beta''_6$ by f'. We also denote by

$$\alpha'_{ij} \quad (i = 1, \dots, 6,\ j = 1, 2)$$

the strict transform on A (and on A') of α_{ij}.

Lemma 4.8. (1) *Let l be a line on B intersecting C. Assume that l is not a bi-secant line of C and the strict transform l' of l on A intersects a flopping curve β'_i. Then $l = \alpha_{ij}$ $(j = 1, 2)$ as in Condition* 3.20(h) *and α_{ij} do not intersect β_k $(k \neq i)$.*

(2) *The curves α'_{ij} on A' $(i = 1, \dots, 6,\ j = 1, 2)$ are fibers of f' intersecting flopped curves, and vice versa.*

(3) *$\widehat{\beta}_i$ is a bi-secant line of $\widehat{C}$ that intersects $\widehat{C}$ transversely at the images of α'_{i1} and α'_{i2}.*

Proof. We follow the notation of Theorem 4.6. First note that any α_{ij} satisfies the assumptions of l in the statement.

Since $H \cdot l' = 1$ and $E \cdot l' = 1$ on A, we have $L \cdot l' = 1$ on A by (4.3). Since $l' \cap \beta'_i \neq \emptyset$ and $L \cdot \beta'_i < 0$, we have $L \cdot l' \leq 0$ on A' by Proposition 4.5. Since L is nef on A', we have $L \cdot l' = 0$, namely, l' is a fiber of f'. Since $H \cdot \beta'_i = 1$ and $-K_A \cdot \beta'_i = 0$ on A, we have $H \cdot \beta''_i = -1$ and $-K_{A'} \cdot \beta''_i = 0$ on A' by Proposition 4.2. Therefore, $L \cdot \beta''_i = 1$ on A' by (4.3), namely, $\widehat{\beta}_i$ is a line on B. If l is different from α_{i1} and α_{i2}, then there are at least three non-trivial fibers of f' intersecting β''_i, and hence $\widehat{\beta}_i$ is at least a tri-secant line of $\widehat{C}$. This is a contradiction, since $\widehat{C}$ is a normal rational curve by Theorem 4.6 (cf. proof of Proposition 3.2). Therefore, $l = \alpha_{ij}$ and $\widehat{\beta}_i$ is a bi-secant line of $\widehat{C}$ that intersects $\widehat{C}$ transversely at the images of α'_{i1} and α'_{i2}.

By the above argument, if an α_{ij} intersects a β_k $(k \neq i)$, then $L \cdot \alpha'_{ij} < 0$ on A', a contradiction. Therefore, any α_{ij} does not intersect β_k $(k \neq i)$.

Let γ be a non-trivial fiber of f'. Then it holds that $-K_{A'} \cdot \gamma = 1$ and $L \cdot \gamma = 0$. Assume that γ intersects some flopped curve on A'. Then, by applying Proposition 4.5 to the flop $A' \dashrightarrow A$, we have $-K_A \cdot \gamma = 1$ and $L \cdot \gamma \geq 1$ on A since the intersection number between L and a flopped curve on A' is positive. Thus, we have $H \cdot \gamma \leq 1$ and $E \cdot \gamma \leq 1$ on A. If $H \cdot \gamma = 1$, then the image of γ by f satisfies the assumptions of l in the statement, which completes the proof. If $H \cdot \gamma = 0$, then $E \cdot \gamma = -1$, namely, γ is a fiber of $E \to C$. In this case, $L \cdot \gamma = 2$ on A. However, by Condition 3.20(g), γ can intersect only one strict transform of a bi-secant line at one point transversely, and hence we have $L \cdot \gamma = 2 - 1 = 1$ on A' by applying Proposition 4.5 to the flop $A \dashrightarrow A'$, a contradiction. □

The following claim shows that the strict transforms on B on the right-hand side of (4.2) of ζ_{i1} and ζ_{i2} play the same role for $\widehat{C}$ as that of α_{i1} and α_{i2} for C (Notation 3.12).

Claim 4.9. *For $i = 1, \dots, 6$, let $\widehat{\zeta}_{i1}$ and $\widehat{\zeta}_{i2}$ be the strict transforms on B on the right-hand side of* (4.2) *of ζ_{i1} and ζ_{i2}. Then $\widehat{\zeta}_{i1}$ and $\widehat{\zeta}_{i2}$ are lines intersecting both $\widehat{C}$ and $\widehat{\beta}_i$ outside $\widehat{C} \cap \hat{\beta}_i$. In particular, Condition* 3.20(h) *holds for $\widehat{C}$.*

Proof. Similarly to the proof of Lemma 4.8, we see that $\widehat{\zeta}_{i1}$ and $\widehat{\zeta}_{i2}$ are lines intersecting $\widehat{C}$ by Proposition 4.5. By Condition 3.20(j) for C, Corollary 4.7 and Lemma 4.8, the strict transforms on B on the right-hand side of (4.2) of γ_{ij1} and γ_{ij2} $(j = 1, 2)$ are the lines through $\widehat{C} \cap \hat{\beta}_i$. Thus, $\widehat{\zeta}_{i1}$ and $\widehat{\zeta}_{i2}$ intersect $\widehat{C}$ outside $\hat{\beta}_i$. □

Claim 4.10. *$\widehat{C}$ satisfies Condition* 3.20(a), (b), (f), (g), (i) *and* (j).

Proof. (a) $\widehat{C}$ is a sextic normal rational curve by Theorem 4.6.

(b) $\widehat{C}$ has only six bi-secant lines $\hat{\beta}_1, \dots, \hat{\beta}_6$, which are the images of the flopped curves $\beta''_1, \dots, \beta''_6 \subset A'$ (see the last assertion of Theorem 4.6).

(f) By Theorem 4.6, the flop $A \dashrightarrow A'$ is Atiyah's flop. Therefore, condition (f) follows by the symmetry of Atiyah's flop.

(g) By Proposition 3.17 and the symmetry of Atiyah's flop, $\beta_1'', \dots, \beta_6''$ are disjoint. We only have to show that any two of them, say β_1'' and β_2'', do not intersect the same fiber of $E' \to \widehat{C}$. Assume by contradiction that β_1'' and β_2'' intersect a fiber γ of $E' \to \widehat{C}$. Then, by (4.3) and Proposition 4.5, we have $H \cdot \gamma = 2 - 2 \times 1 = 0$ on A. Therefore, γ on A is a fiber of f, and hence $\beta_1 \cap \beta_2 \neq \emptyset$, a contradiction.

(i) This follows from (j) for C and Lemma 4.8.

(j) Fix one p_{ij} (Notation 3.12). By (i) for C, there are two lines r_1 and r_2 different from β_i through p_{ij}. We denote by r_1' and r_2' the strict transforms on A of r_1 and r_2, respectively. By Lemma 4.8, r_1' and r_2' are disjoint from any flopping curves. Therefore, r_1' and r_2' intersect ζ_{ij} on A' and, by the proof of Corollary 4.7, the images $\widehat{r}_1$ and $\widehat{r}_2$ by f' of r_1' and r_2', respectively, are lines intersecting $\widehat{C}$. Note that by Claim 4.9, $\widehat{\zeta}_{ij}$ plays the same role for $\widehat{C}$ as that of one of α_{ij} for C. Therefore, $\widehat{r}_1$ and $\widehat{r}_2$ play the same role for $\widehat{\zeta}_{ij}$ as that of γ_{ij1} and γ_{ij2} for α_{ij}. □

Lemma 4.11. *A line on A intersecting one β_i' is one of the following (a similar statement holds for A'):*

(1) *the strict transform α_{ij}' of α_{ij} $(1 \le j \le 2)$, or*
(2) *the line l_{ij} $(1 \le j \le 2)$ as in Proposition 3.14(2).*

Proof. Fix one β_i. By Lemma 4.8, the strict transform intersecting β_i' of a line intersecting C is α_{ij}' $(1 \le j \le 2)$ or β_i' itself. Thus, the assertion follows for A by Proposition 3.14.

Since we have already checked Conditions 3.20(g) and (h) for $\widehat{C}$, the assertion also holds for A'. □

Lemma 4.12. *There exists a natural one-to-one correspondence between lines on A and lines on A' as follows:*

(1) *For a line on A disjoint from $\beta_1', \dots, \beta_6'$, its strict transform on A' is a line on A' disjoint from $\beta_1'', \dots, \beta_6''$, and vice versa.*
(2) *Fix one β_i. A line on A intersecting β_i' and a line on A' intersecting β_i'' correspond to each other as follows (Lemma 4.11):*
 (2-1) *The line α_{ij}' $(1 \le j \le 2)$ on A corresponds to the line $\alpha_{ij}' \cup \beta_i''$ on A'.*
 (2-2) *The line l_{ij} $(1 \le j \le 2)$ on A corresponds to ζ_{ij} on A' (Notation 3.12 and Proposition 3.14).*

Proof. Let l be a line on A. Assertion (1) follows from Corollary 4.7.

Assume that l intersects some flopping curve β_i' of $A \dashrightarrow A'$. By Lemma 4.11, there are two cases:

(a) $l = \alpha_{ij}'$. Then, by Lemma 4.8, l is a fiber of $E' \to \widehat{C}$ on A'. Moreover, l intersects the flopped curve β_i''. Hence, the union $l \cup \beta_i''$ is a line on A' of type (ii) as in Proposition 3.14.
 In this case, l corresponds to the line $l \cup \beta_i''$ on A'.
(b) l is the union of one β_i' and a fiber ζ_{ij} of E over one point p_{ij} of $C \cap \beta_i$ (Notation 3.12).
 This case is reduced to case (a) by exchanging the role of A and A' (note also Claim 4.9 and Lemma 4.11).
 In this case, l corresponds to the line ζ_{ij} on A'.

Thus, in any case, a line on A corresponds to the unique line on A' and vice versa. □

We can define $\widehat{\mathcal{H}}_1 := \varphi^{-1}(\widehat{C})$ as in (3.1), which is the triple cover of $\widehat{C}$. By Proposition 3.14 for A' and the proof of Proposition 3.15, we see that $\widehat{\mathcal{H}}_1$ parameterizes lines on A'.

Claim 4.13. *$\widehat{C}$ satisfies Condition 3.20(c). Moreover, $\widehat{\mathcal{H}}_1$ is isomorphic to $\mathcal{H}_1$.*

Proof. By Lemma 4.12, there exists a natural homeomorphism $\mathcal{H}_1 \to \widehat{\mathcal{H}}_1$. In particular, $\widehat{\mathcal{H}}_1$ is irreducible. By Condition 3.20(b) for $\widehat{C}$, the morphism $\widehat{\mathcal{H}}_1 \to M(\widehat{C})$ is birational and $\deg M(\widehat{C}) = 6$. By Lemma 4.8, each $\hat{\beta}_i$ intersects $\widehat{C}$ at two points, and hence the inverse image of each $[\hat{\beta}_i]$ by $\widehat{\mathcal{H}}_1 \to M(\widehat{C})$ consists of two points. Thus, $p_a(\widehat{\mathcal{H}}_1) \le 4$. Therefore, $\widehat{\mathcal{H}}_1$ is smooth and the above homeomorphism is an isomorphism. Moreover, $[\hat{\beta}_1], \dots, [\hat{\beta}_6]$ are simple nodes of $M(\widehat{C})$ and $M(\widehat{C})$ has no other singularities. □

Lemma 4.14. *Let g_i be the unique conic on $\mathbb{P}^2$ passing through $[\beta_1], \dots, [\check{\beta}_i], \dots, [\beta_6]$ (Condition 3.20(d)). Then $[\alpha_{i1}], [\alpha_{i2}]$ are precisely the intersection points of $M(\beta_i)$ and g_i.*

Proof. It suffices to show that $[\alpha_{i1}], [\alpha_{i2}]$ are contained in g_i because we already know that they are contained in $M(\beta_i)$.

Let α be any line on B intersecting β_i. Since $M(C) \cdot M(\alpha) = 6$, the subset $M(C) \cap M(\alpha)$ consists of $[\beta_i]$ and four points $[\gamma_1], \dots, [\gamma_4]$ (note that $[\beta_i]$ is a node of $M(C)$). If we move α, then the line $M(\alpha)$ moves in the pencil of lines through $[\beta_i]$. Therefore, if α is general, then the four lines $\gamma_1, \dots, \gamma_4$ are mutually distinct and are different from β_i. Let $\gamma_1', \dots, \gamma_4'$ be the strict transforms on A of $\gamma_1, \dots, \gamma_4$. They are lines on A. Then it holds that

$$[\gamma_1'] + \cdots + [\gamma_4'] = (\pi_{|\mathcal{H}_1})^*(M(\alpha)_{|M(C)}) - [l_{i1}] - [l_{i2}], \tag{4.4}$$

where l_{ij} are lines on A as in Proposition 3.14(ii). Let $\alpha_{i1}', \alpha_{i2}'$ be the strict transforms of α_{i1} and α_{i2}, which are lines on A. Then it holds that

$$[\alpha_{i1}'] + [\alpha_{i2}'] \sim (\pi_{|\mathcal{H}_1})^*(M(\beta_i)_{|M(C)}) - (\delta - [l_{i1}]) - (\delta - [l_{i2}]), \tag{4.5}$$

where we recall that δ is the g^1_3 defining the triple cover $\mathcal{H}_1 \to C$ (Section 3.3.3). Let r_{jk} $(1 \le j \le 2, 1 \le k \le 2)$ be the lines through p_{ij} different from β_i, and let r_{jk}' be the strict transform of r_{jk} on A (Notation 3.12). Then $M(C) \cap M(\beta_i)$ consists of six points $[\alpha_{i1}], [\alpha_{i2}], [r_{11}], [r_{12}], [r_{21}], [r_{22}]$, and $[r_{j1}'] + [r_{j2}'] + [l_{ij}] \sim \delta$ $(1 \le j \le 2)$. Thus, we have (4.5).

Summing (4.4) and (4.5), we obtain

$$[\alpha_{i1}'] + [\alpha_{i2}'] + [\gamma_1'] + \cdots + [\gamma_4'] \sim (\pi_{|\mathcal{H}_1})^*\mathcal{O}_{M(C)}(2) - 2\delta = 2\theta \sim K_{\mathcal{H}_1}. \tag{4.6}$$

By Condition 3.20(d) for C, we know that $[\beta_1], \dots, [\beta_6]$ are in a general position. Let S be the cubic surface obtained by the blow-up of $\mathbb{P}^2$ at $[\beta_1], \dots, [\beta_6]$, let $\lambda \subset S$ be the total transform of a line on $\mathbb{P}^2$ and let ε_i be the exceptional curve over the point $[\beta_i]$. For $\mathcal{H}_1 \subset S$ it holds that

$$K_{\mathcal{H}_1} \sim \left(3\lambda - \sum_{j=1}^{6} \varepsilon_j\right)_{|\mathcal{H}_1}. \tag{4.7}$$

Since $\varepsilon_{i|\mathcal{H}_1} = [l_{i1}] + [l_{i2}]$, equality (4.4) implies that

$$[\gamma_1'] + \cdots + [\gamma_4'] \sim (\lambda - \varepsilon_i)_{|\mathcal{H}_1}.$$

Thus, by (4.6) and (4.7), it holds that

$$[\alpha_{i1}'] + [\alpha_{i2}'] \sim \left\{\left(3\lambda - \sum_{j=1}^{6} \varepsilon_j\right) - (\lambda - \varepsilon_i)\right\}_{|\mathcal{H}_1} = \{2\lambda - (\varepsilon_1 + \cdots + \check{\varepsilon}_i + \cdots \varepsilon_6)\}_{|\mathcal{H}_1}.$$

Since $\mathcal{H}_1$ is not hyperelliptic, $[\alpha'_{i1}] + [\alpha'_{i2}]$ does not move, and thus $[\alpha'_{i1}] + [\alpha'_{i2}]$ is cut out by the strict transform of the conic g_i. Therefore, $[\alpha_{i1}]$, $[\alpha_{i2}]$ are contained in g_i, which completes the proof. □

Claim 4.15. *$\widehat{C}$ satisfies Conditions 3.20*(d) *and* (e).
Therefore, together with Claims 4.9, 4.10 and 4.13, we have $[\widehat{C}] \in \mathcal{H}^\circ$.

Proof. (d) By Claim 4.13, we can identify $\mathcal{H}_1$ and $\widehat{\mathcal{H}}_1$, and then we have a map $\pi_{\widehat{C}}: \mathcal{H}_1 \simeq \widehat{\mathcal{H}}_1 \to M(\widehat{C})$. We show that $[\alpha'_{i1}], [\alpha'_{i2}] \in \mathcal{H}_1$ are mapped by $\pi_{\widehat{C}}$ to the node $[\hat{\beta}_i] \in M(\widehat{C})$ (notation as in Lemma 4.11). For simplicity, we denote the strict transforms on A' of curves and divisors on A using the same notation. On A', α'_{i1} and α'_{i2} are the fibers of E' through $E' \cap \beta''_i$ by Lemma 4.8. Thus, by applying Corollary 3.16 to $\widehat{C}$, the two lines $\alpha'_{i1} \cup \beta''_i$ and $\alpha'_{i2} \cup \beta''_i$ on A' correspond to the node $[\widehat{\beta}_i]$ of $M(\widehat{C})$. By Lemma 4.12, the two lines $\alpha'_{i1} \cup \beta''_i$ and $\alpha'_{i2} \cup \beta''_i$ on A' correspond to the two lines α'_{i1} and α'_{i2} on A. Therefore, $[\alpha'_{i1}], [\alpha'_{i2}] \in \mathcal{H}_1$ are mapped to the node $[\hat{\beta}_i] \in M(\widehat{C})$ by $\pi_{\widehat{C}}$.

Now we consider $\mathcal{H}_1$ contained in the cubic surface S obtained by the blow-up of $\mathbb{P}^2$ at $[\beta_1], \dots, [\beta_6]$, which are in a general position by (d) for C. Let g'_i be the strict transform of g_i as in Lemma 4.14. Then $\mathcal{H}_1 \to M(\widehat{C})$ is the restriction of the contraction $S \to \mathbb{P}^2$ of $g'_1, \dots, g'_6$ since $[\alpha'_{i1}] + [\alpha'_{i2}]$ on $\mathcal{H}_1$ is the restriction of g'_i by the proof of Lemma 4.14. In particular, $[\hat{\beta}_1], \dots, [\hat{\beta}_6]$ are in a general position.

(e) As in the verification of (d) for $\widehat{C}$, we identify $\mathcal{H}_1$ and $\widehat{\mathcal{H}}_1$. Thus, $\widehat{\mathcal{H}}_1$ has the theta characteristic θ. Let δ' be the g^1_3 of $\mathcal{H}_1$ such that $\delta + \delta' = K_{\mathcal{H}_1}$. Now we show that

$$\pi^*_{\widehat{C}} \mathcal{O}_{\mathbb{P}^2}(1)_{|M(\widehat{C})} = \delta' + \theta. \tag{4.8}$$

We use the notation as in the proof of Lemma 4.14. Note that $(\delta' + \theta) + (\delta + \theta) = (\delta' + \delta) + 2\theta = 2K_{\mathcal{H}_1}$. Therefore, $\delta' + \theta = 2(3\lambda - \sum_{i=1}^6 \varepsilon_i) - \lambda = 5\lambda - 2\sum_{i=1}^6 \varepsilon_i$. On the other hand, by the verification of (d) for $\widehat{C}$, $\mathcal{H}_1 \to M(\widehat{C})$ is the restriction of the contraction $S \to \mathbb{P}^2$ of $g'_1, \dots, g'_6$, and thus we have $\pi^*_{\widehat{C}} \mathcal{O}_{\mathbb{P}^2}(1)_{|M(\widehat{C})} = 5\lambda - 2\sum_{i=1}^6 \varepsilon_i$. Hence, (4.8) holds.

By the correspondence of lines on A and lines on A', the theta characteristics on $\mathcal{H}_1$ and $\widehat{\mathcal{H}}_1$ have the same meaning. Therefore, by (4.8), δ' is associated with the triple cover $\widehat{\mathcal{H}}_1 \to \widehat{C}$. This completes the proof. □

Remark. Note that $\mathcal{H}_1$ is a general curve of genus four for a general C by [30, Corollary 4.0.6]. Therefore, in the verification of (e) for $\widehat{C}$, we have $\delta \neq \delta'$ for a general C.

As reviewed in the Introduction, the natural rational map $\pi_{\mathcal{S}_4^+}: \mathcal{H} \dashrightarrow \mathcal{S}_4^+$, $C \mapsto (\mathcal{H}_1, \theta)$ is the composite of the rational maps $p_{\mathcal{S}_4^+}: \mathcal{H} \dashrightarrow \widetilde{\mathcal{S}}_4^+$ and $q_{\mathcal{S}_4^+}: \widetilde{\mathcal{S}}_4^+ \dashrightarrow \mathcal{S}_4^+$, where a general fiber of $p_{\mathcal{S}_4^+}$ is a G-orbit in $\mathcal{H}$ and $q_{\mathcal{S}_4^+}$ is birational or of degree two. We can now refine this description as follows.

Corollary 4.16. *The rational map $q_{\mathcal{S}_4^+}$ is of degree two. The covering transformation of $q_{\mathcal{S}_4^+}$ exchanges the classes of C and $\widehat{C}$ on $\widetilde{\mathcal{S}}_4^+$.*

Proof. Let $[C]$ be a general point of $\mathcal{H}$. Since $[C]$ and $[\widehat{C}]$ are mapped to the same $[(\mathcal{H}_1, \theta)] \in \mathcal{S}_4^+$, we only have to show that $[C]$ and $[\widehat{C}]$ are not G-equivalent. This follows since $\delta \neq \delta'$ by the remark after the proof of Claim 4.15. □

Let

$$\widetilde{\mathcal{S}}_4^{+\circ} \subset \widetilde{\mathcal{S}}_4^{+}$$

be the image of $\mathcal{H}^\circ$ and let

$$\mathcal{S}_4^{+\circ} \subset \mathcal{S}_4^{+}$$

be the image of $\widetilde{\mathcal{S}}_4^{+\circ}$. By Corollary 4.16, we have an involution on $\widetilde{\mathcal{S}}_4^{+\circ}$, which is the deck transformation of the double cover $\widetilde{\mathcal{S}}_4^{+\circ} \to \mathcal{S}_4^{+\circ}$.

5. Birational model of $\mathcal{S}_4^+$

In this section, we construct a birational model of $\mathcal{S}_4^+$.

5.1. $\mathcal{H}$ is birational to $(\mathbb{P}^2)^6/\mathfrak{S}_6$

We recall that $\mathcal{H}$ is the closure of the Hilbert scheme of sextic normal rational curves on B, and $G = \mathrm{PGL}_2$. By Proposition 3.7, $\mathcal{H}$ is an irreducible 12-dimensional variety. The G-action on B induces the G-action on $\mathcal{H}$. Let

$$\mathcal{H}^* \subset \mathcal{H}$$

be the open subset consisting of (reduced but possibly reducible) sextic curves with exactly six different bi-secant lines. Then we define a G-equivariant morphism

$$\Theta\colon \mathcal{H}^* \to (\mathbb{P}^2)^6/\mathfrak{S}_6, \qquad C \mapsto ([\beta_1], \dots, [\beta_6]).$$

To construct a birational model of $\mathcal{S}_4^+$, the following theorem is crucial. Mori theory for threefolds plays an important role in its proof.

Theorem 5.1. *The morphism Θ is birational. Moreover, $\Theta_{|\mathcal{H}^\circ}$ is an isomorphism onto its image.*

Proof. Since $\dim \mathcal{H} = \dim(\mathbb{P}^2)^6/\mathfrak{S}_6 = 12$, it suffices to show that $\Theta_{|\mathcal{H}^\circ}$ is an isomorphism onto its image. Since $(\mathbb{P}^2)^6/\mathfrak{S}_6$ is normal, we only have to show that $\Theta_{|\mathcal{H}^\circ}$ is injective by the main Zariski theorem. By contradiction, assume that there exists $[C]$, $[\Gamma] \in \mathcal{H}^\circ$ such that $C \neq \Gamma$ and $\beta_1 \dots, \beta_6$ are bi-secant lines of both C and Γ. Let α_{ij} $(1 \le i \le 6, 1 \le j \le 2)$ be the lines associated with C as in Condition 3.20(h). Then, by Lemma 4.14, α_{ij} have the same meaning for Γ. We consider the diagram (4.2) in Theorem 4.6 for C and we use the notation there freely. Let Γ' be the strict transform of Γ on A. For simplicity, we denote by the same symbol the strict transforms on A and A' of curves on B. For B on the right-hand side of (4.2), let $\widehat{\Gamma}$ be the strict transform of Γ and let $\hat{\beta}_i$ be the image of the flopped curve corresponding to β_i.

We show that $\deg \widehat{\Gamma} \le 6$. We define the non-negative integer a by the equation $-K_A \cdot \Gamma' = 12 - a$; equivalently, C intersects Γ on B on the left-hand side of (4.2) at a points counted with multiplicities. Since Γ intersects $\beta_1 \cup \dots \cup \beta_6$ at 12 points, Γ' intersects $\beta_1' \cup \dots \cup \beta_6'$ at $12 - a$ or more points, depending on the common intersection points of C, Γ and β_i. This implies that $H \cdot \Gamma' \ge 6 + 12 - a$ on A' by Proposition 4.5. By (4.3) in Theorem 4.6, we have $L \cdot \Gamma' \le 2(12 - a) - (18 - a) = 6 - a \le 6$ on A'. Thus, $\deg \widehat{\Gamma} \le 6$.

Since $\deg \Gamma = 6$, we have $H \cdot \Gamma' = 6$ on A. By Proposition 4.4(2), it holds that $H \cdot \Gamma' \ge 6$ on A'. Since L is nef on A' and Γ' is not a fiber of $A' \to B$ by Lemma 4.8, it holds that $L \cdot \Gamma' \ge 1$.

Thus, it holds that $-K_{A'} \cdot \Gamma' \geq 4$ on A' by (4.3) as in Theorem 4.6. By Proposition 4.4(1), it holds that $-K_A \cdot \Gamma' \geq 4$ on A. On the other hand, $-K_B \cdot \Gamma = 12$ on B on the left-hand side in diagram (4.2). Therefore, since $-K_A = f^*(-K_B) - E$, we see that Γ intersects C at eight or fewer points counted with multiplicities. Thus, by the pigeonhole principle, for at least two bi-secant lines of C, say β_1 and β_2, Γ passes through at most one of $p_{11}, p_{12}, t_{11}, t_{12}$ and one of $p_{21}, p_{22}, t_{21}, t_{22}$, where $t_{ij} := C \cap \alpha_{ij}$ $(i = 1, 2,\ j = 1, 2)$ (Notation 3.12). This implies that $\hat{\beta}_1$ and $\hat{\beta}_2$ are at least tri-secant lines of $\widehat{\Gamma}$ since α'_{ij} on A' is a fiber of f' intersecting $\hat{\beta}_i$ by Lemma 4.8 (if $\hat{\beta}_i$ passes through a singular point of $\widehat{\Gamma}$, then we regard $\hat{\beta}_i$ as a multi-secant line of $\widehat{\Gamma}$).

Consider the projection $B \dashrightarrow Q$ from the line $\hat{\beta}_1$ (Proposition 2.6). Then the degree of the image $\widehat{\Gamma}'$ of $\widehat{\Gamma}$ is at most three since $\deg \widehat{\Gamma} \leq 6$ and $\hat{\beta}_1$ is at least a tri-secant line of $\widehat{\Gamma}$. It holds that $\hat{\beta}_1 \cap \hat{\beta}_2 = \emptyset$ since $\widehat{C}$ satisfies Condition 3.20(g) (proof of Claim 4.10). Thus, the image of $\hat{\beta}_2$ on Q is at least tri-secant lines of $\widehat{\Gamma}'$. This, however, is impossible. Indeed, if $\deg \widehat{\Gamma}' = 1, 2$, then this is obvious. If $\deg \widehat{\Gamma}' = 3$, then $\widehat{\Gamma}'$ is a twisted cubic curve since a plane cubic curve does not exist on Q. Thus, again $\widehat{\Gamma}'$ cannot have a tri-secant line. $\square$

5.2. *Birational model of $\mathcal{S}_4^+$*

Let

$$U \subset (\mathbb{P}^2)^6$$

be the set of stable ordered six points with respect to the symmetric linearization of the action of PGL_3 or, more explicitly, the set of ordered six points such that no two points coincide, or no four points are collinear [6, Theorem 1]. From this explicit description, we see that U is $\mathfrak{S}_6$-invariant. Note that the geometric quotient U/G exists. Let $\mathcal{L}$ be the restriction of the PGL_3-linearized line bundle to U. By restricting the PGL_3-action to the G-action, $\mathcal{L}$ is also G-linearized. We claim that U consists of the set of G-stable points. Let $x \in U$ be a point. The stabilizer group of x for the G-action is finite (actually trivial) since it is so for the PGL_3-action. There exists a PGL_3-invariant section s of some multiple of $\mathcal{L}$ such that $s(x) \neq 0$ and $\mathrm{PGL}_3 \cdot x$ is closed in $U_s := \{y \in U \mid s(y) \neq 0\}$. Since $G \subset \mathrm{PGL}_3$ is a closed subgroup, the same is true for G.

Set

$$V = U/\mathfrak{S}_6 \subset (\mathbb{P}^2)^6/\mathfrak{S}_6.$$

Since the G-action and $\mathfrak{S}_6$-action commute, V/G also exists and $V/G \simeq (U/G)/\mathfrak{S}_6$.

Let V_1 be the image of $\mathcal{H}^\circ$ on $(\mathbb{P}^2)^6/\mathfrak{S}_6$. Note that $V_1 \subset V$ by Condition 3.20(d).

Then by Corollary 4.16 and Theorem 5.1, the involution associated with the map $\widetilde{\mathcal{S}}_4^{+\circ} \to \mathcal{S}_4^{+\circ}$ is translated to an involution J on V_1/G satisfying

$$J : \Theta([C]) \mapsto \Theta([\widehat{C}]).$$

We can sum up the above discussion as follows.

Proposition 5.2. *$\mathcal{S}_4^+$ is birational to $(V_1/G)/J$.*

6. Rationality proof of $\mathcal{S}_4^+$

By Proposition 5.2, it suffices to show $(V_1/G)/J$ is rational. For this purpose, we investigate the variety $(V_1/G)/J$ relating it to the following classically well-studied variety:

$$X := (\mathbb{P}^2)^6//\mathrm{PGL}_3,$$

where the GIT-quotient is taken with respect to the symmetric linearization of the action of PGL_3 [6, Proposition 1]. This is a compactification of the moduli space of ordered six points on $\mathbb{P}^2$. By [6, Example 3], X is isomorphic to the quartic hypersurface in $\mathbb{P}(1^5, 2)$. Let

$$Y := X/\mathfrak{S}_6.$$

Note that there exists a natural morphism

$$\varrho : V_1/G \to Y$$

since $V_1 \subset (\mathbb{P}^2)^6/\mathfrak{S}_6$ and the G-action on $(\mathbb{P}^2)^6$ commutes with the $\mathfrak{S}_6$-action on $(\mathbb{P}^2)^6$.

6.1. J is a lifting of the association map

We show that J is a lifting of the classical association map on Y.

By [6, Example 4], there exists an involution j' on X called the *(ordered) association map*. We do not provide a definition of j' but only describe it for the open subset of X that parameterizes ordered six points in general positions [6, pp. 118–120].

Let $\Sigma \subset \mathbb{P}^3$ be a smooth cubic surface and let $\sigma \colon \Sigma \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at six points $p_1, \ldots, p_6$. We consider ordered sets of six lines on Σ or, equivalently, ordered sets of six points on $\mathbb{P}^2$, whereas up to now we have considered only unordered sets of six points on $\mathbb{P}^2$. The 27 lines on Σ can be grouped into three ordered subsets:

$$(l_1, \ldots, l_6), (l'_1, \ldots, l'_6), (m_{ij}) \quad (1 \le i < j \le 6),$$

where the lines l_i are the exceptional lines $\sigma^{-1}(p_i)$, the lines l'_i are the strict transforms of the conics $q_i \subset \mathbb{P}^2$ passing through $p_1, \ldots, \check{p}_i, \ldots, p_6$, and the lines m_{ij} are the strict transforms of the lines $\langle p_i, p_j \rangle$ joining the points p_i and p_j. The first two groups of lines, $(l_1, \ldots, l_6)$ and $(l'_1, \ldots, l'_6)$, form a *double sixer*, which means that

$$l_j \cap l_j = \emptyset, \qquad l'_i \cap l'_j = \emptyset, \qquad l_i \cap l'_j \ne \emptyset \quad \text{if and only if} \quad i \ne j.$$

Every set of six disjoint lines on Σ can be included in a unique double sixer, from which Σ can be reconstructed uniquely. There are 36 double sixers of Σ. Every double sixer defines two regular birational maps $\sigma : \Sigma \to \mathbb{P}^2, \sigma' : \Sigma \to \mathbb{P}^2$, each of which blows down one of the two sixes (sextuples of disjoint lines) of the double sixers. The association map j' interchanges the two collections of ordered six points in $\mathbb{P}^2$ given by $(\sigma(l_1), \ldots, \sigma(l_6))$ and $(\sigma'(l'_1), \ldots, \sigma'(l'_6))$; namely, it holds that

$$j' \colon (\sigma(l_1), \ldots, \sigma(l_6)) \mapsto (\sigma'(l'_1), \ldots, \sigma'(l'_6)).$$

We also note that j' fixes any ordered sextuple of points lying on a conic.

Since the symmetric group $\mathfrak{S}_6$ acts on the quotient X and its action commutes with j', the map j' descends to an involution j on $X/\mathfrak{S}_6$. The map j is called the *(unordered) association map*.

Proposition 6.1. *The involution J is a lifting of j.*

Proof. The assertion follows from the verification of Condition 3.20(d) in Claim 4.15 and the description of the association map as above. $\square$

6.2. Rationality of the moduli space of double sixers on $\mathbb{P}^2$

The rationality of Y/j was known classically and proved by Coble, and was reproved by Dolgachev [4, Appendix].

Theorem 6.2 (*Coble and Dolgachev*). *The quotient variety Y/j is a rational variety. More explicitly, Y is a hypersurface of degree* 34 *in* $\mathbb{P}(2,3,4,5,6,17)$ *and* $Y/j \simeq \mathbb{P}(2,3,4,5,6)$.

Remark. This result is subtle; it is not known whether the moduli space Y of unordered six points on $\mathbb{P}^2$ is rational or not.

6.3. Quasi $\mathbb{P}^5$-fibration structure

The following diagram summarizes our construction above:

$$\begin{tikzcd} & \mathcal{H}^\circ \arrow[r, "p^\circ"] \arrow[d, "\Theta|_{\mathcal{H}^\circ}"', "\text{isom.}"] & \widetilde{\mathcal{S}}_4^{+\circ} \arrow[r, "q^\circ"] \arrow[d, "\text{isom.}"] & \mathcal{S}_4^{+\circ} \arrow[d, "\text{isom.}"] \\ V & V_1 \arrow[l, hook'] \arrow[r] & V_1/G \arrow[r] \arrow[d, "\varrho"'] & (V_1/G)/J \arrow[d] \\ & & Y \arrow[r] & Y/j, \end{tikzcd} \tag{6.1}$$

where p° is the restriction of $p_{\mathcal{S}_4^+}$ to $\mathcal{H}^\circ$ and q° is the restriction of $q_{\mathcal{S}_4^+}$ to $\widetilde{\mathcal{S}}_4^{+\circ}$.

We consider the following diagram:

$$\begin{tikzcd} U \arrow[r] \arrow[d, "\widehat{\pi}_{\mathrm{PGL}_3}"'] & V \arrow[d, "\pi_{\mathrm{PGL}_3}"'] \\ U/\mathrm{PGL}_3 \arrow[r, "h"] \arrow[d, hook] & (U/\mathrm{PGL}_3)/\mathfrak{S}_6 \arrow[d, hook] \\ X \arrow[r] & Y, \end{tikzcd} \tag{6.2}$$

where all the horizontal arrows correspond to the quotients by $\mathfrak{S}_6$.

Let

$$X_1 \subset U/\mathrm{PGL}_3$$

be the open subset where the morphism $h : U/\mathrm{PGL}_3 \to (U/\mathrm{PGL}_3)/\mathfrak{S}_6$ is étale, and let

$$Y_1 := h(X_1).$$

Lemma 6.3. *The map π_{PGL_3} restricts to a principal fiber bundle with group* PGL_3 *over* Y_1.

Proof. By [6, proof of Theorem 2], $\widehat{\pi}_{\mathrm{PGL}_3}$ is a principal fiber bundle of PGL_3. In addition, X is isomorphic to a quartic hypersurface in $\mathbb{P}(1^5, 2)$ [6, Example 3], and hence its degree $\mathcal{O}_X(1)^4$ is equal to 2. Since Y is a hypersurface of degree 34 in $\mathbb{P}(2,3,4,5,6,17)$ by Theorem 6.2, its degree $\mathcal{O}_Y(1)^4$ is equal to $\frac{2}{6!}$. Therefore, the degree of the map h in diagram (6.2) is 6!, which is equal to the order of $\mathfrak{S}_6$. Hence, $\mathfrak{S}_6$ acts trivially on fibers of $\widehat{\pi}_{\mathrm{PGL}_3}$ over points in the open subset X_1 of U/PGL_3 where h is étale. By [24, Propositions 0.2 and 0.9], π_{PGL_3} is a principal fiber bundle of PGL_3 over $Y_1 := h(X_1)$. $\square$

Claim 6.4. *Each of the open subsets Y_1 and $\varrho(V_1/G)$ of Y is preserved by the involution j.*

Proof. For $\varrho(V_1/G)$, the assertion is clear from the correspondence between J and j. For Y_1, the assertion follows from the description of j in Section 6.1 and the definition of Y_1. $\square$

Let

$$Y_2 := \varrho(V_1/G) \cap Y_1 \quad \text{and} \quad X_2 := h^{-1}(Y_2).$$

Set

$$W := (\pi^{-1}_{\mathrm{PGL}_3}(Y_2)/G) \cap (V_1/G)$$

and denote by

$$\varrho' \colon W \to Y_2$$

the natural morphism. From diagram (6.2) and the proof of Lemma 6.3, we obtain the following diagram:

$$\begin{tikzcd}
\widehat{\pi}^{-1}_{\mathrm{PGL}_3}(X_2)/G \arrow[r] \arrow[d] & \pi^{-1}_{\mathrm{PGL}_3}(Y_2)/G \arrow[d] & W \arrow[l, hook'] \arrow[dl, "\varrho'"] \\
X_2 \arrow[r] \arrow[d, hook] & Y_2 \arrow[d, hook] & \\
X \arrow[r] & Y &
\end{tikzcd} \tag{6.3}$$

By Claim 6.4, Y_2 is preserved by j. Then W is preserved by the involution J. We also denote by J the restriction of J to W.

Lemma 6.5. *Every fiber of ϱ' is an open subset of $\mathbb{P}^5$.*

Proof. A fiber of $\pi^{-1}_{\mathrm{PGL}_3}(Y_2)/G \to Y_2$ is isomorphic to PGL_3/G, which is isomorphic to an open subset of $\mathbb{P}^5$ by Claim 2.3. $\square$

Proposition 6.6. *The morphism ϱ' extends to a $\mathbb{P}^5$-bundle $\overline{\varrho}' \colon \overline{W} \to Y_2$ over Y_2.*

Proof. We only have to extend $\pi^{-1}_{\mathrm{PGL}_3}(Y_2)/G \to Y_2$ to a $\mathbb{P}^5$-bundle. By [6, proof of Theorem 2], $\widehat{\pi}_{\mathrm{PGL}_3}$ is locally trivial in the Zariski topology. Take an open covering $\{X^i\}$ of X_2 such that $\widehat{\pi}^{-1}_{\mathrm{PGL}_3}(X^i) \to X^i$ is isomorphic PGL_3-equivariantly to $\pi^i \colon \mathrm{PGL}_3 \times X^i \to X^i$, where PGL_3 acts on $\mathrm{PGL}_3 \times X^i$ by the left multiplication on the first factor and trivially on the second factor. Denote by ι_i the isomorphism $\widehat{\pi}^{-1}_{\mathrm{PGL}_3}(X^i) \to \mathrm{PGL}_3 \times X^i$. Then $\widehat{\pi}^{-1}_{\mathrm{PGL}_3}(X^i)/G \to X^i$ is isomorphic to $(\mathrm{PGL}_3/G) \times X^i$ and the latter can be compactified to the product $\mathbb{P}^5 \times X^i$ by Claim 2.3. Note that the isomorphism

$$\iota_{ij} := \iota_{j|\widehat{\pi}^{-1}_{\mathrm{PGL}_3}(X^i \cap X^j)} \circ \iota_{i|\widehat{\pi}^{-1}_{\mathrm{PGL}_3}(X^i \cap X^j)}{}^{-1} \colon \mathrm{PGL}_3 \times (X^i \cap X^j) \to \mathrm{PGL}_3 \times (X^i \cap X^j)$$

is PGL_3-equivariant and PGL_3 acts on each of the first factors by the left multiplication. Therefore, it maps (g, x) to (gh, x) for any $g \in \mathrm{PGL}_3$ and $x \in X^i \cap X^j$, where h is the element of PGL_3 such that $\iota_{ij}(\mathrm{id}, x) = (h, x)$. Thus, by choosing the map $\mathrm{PGL}_3 \to \mathbb{P}^5$ as in the last part of Claim 2.3, the morphism ι_{ij} descends to $(\mathrm{PGL}_3/G) \times (X^i \cap X^j) \to (\mathrm{PGL}_3/G) \times (X^i \cap X^j)$ mapping $({}^t gg, x)$ to $({}^t h({}^t gg)h, x)$. Since this is linear on the fiber PGL_3/G, it naturally extends to $\mathbb{P}^5 \times (X^i \cap X^j) \to \mathbb{P}^5 \times (X^i \cap X^j)$ and these extensions patch $\mathbb{P}^5 \times X^i$'s. Thus, we obtain a locally trivial $\mathbb{P}^5$-bundle over X_2 containing $\widehat{\pi}^{-1}_{\mathrm{PGL}_3}(X_2)/G$ as an open subset.

Finally, we show that the $\mathfrak{S}_6$-action on $\widehat{\pi}^{-1}_{\mathrm{PGL}_3}(X_2)/G$ extends to the $\mathbb{P}^5$-bundle, which implies that the quotient by the $\mathfrak{S}_6$-action is the $\mathbb{P}^5$-bundle over Y_2 as desired.

Fix an element τ of $\mathfrak{S}_6$. Since $\mathfrak{S}_6$ acts trivially on the fibers of $\widehat{\pi}_{\mathrm{PGL}_3}$ over points of X_2 and its action commutes with the PGL_3-action, the isomorphism ι_i induces a PGL_3-equivariant isomorphism $\tau\big(\widehat{\pi}^{-1}_{\mathrm{PGL}_3}(X^i)\big) \to \mathrm{PGL}_3 \times \tau(X^i)$. Thus, we can assume that $\tau(X^i)$ is included in the open covering $\{X^i\}$ for every i, namely, $\tau(X^i) = X^{i'}$ for some i'. Since the isomorphism $\iota_{i'} \circ \tau \circ \iota_i^{-1} \colon \mathrm{PGL}_3 \times X^i \to \mathrm{PGL}_3 \times X^{i'}$ is PGL_3-equivariant, it is the right multiplication of some $h_i \in \mathrm{PGL}_3$ on the first factor. Thus, as in the argument above, this descends to an isomorphism $(\mathrm{PGL}_3/G) \times X^i \to (\mathrm{PGL}_3/G) \times X^{i'}$ and extends to an isomorphism $\mathbb{P}^5 \times X^i \to \mathbb{P}^5 \times X^{i'}$. These extensions patch and give the action of τ on the $\mathbb{P}^5$-bundle. □

6.4. *Quasi $\mathbb{P}^4$-subfibration*

We look for a subfibration of $\pi^{-1}_{\mathrm{PGL}_3}(Y_2)/G \to Y_2$ whose fiber is an open subset of a hyperplane of $\mathbb{P}^5$.

Let

$$D' \subset (\mathbb{P}^2)^6/\mathfrak{S}_6$$

be the closure of the set of unordered six points, two of which are polar with respect to $\widetilde{\Omega}$ (Proposition 2.2). By definition, D' is G-invariant.

Lemma 6.7. *The locus D' is a prime divisor of $(\mathbb{P}^2)^6/\mathfrak{S}_6$. For a general point $([l_1], \dots, [l_6]) \in D'$, it holds that*

(1) *only two of six lines $l_1, \dots, l_6$ intersect on B;*
(2) *six points $[l_1], \dots, [l_6] \in \mathbb{P}^2$ are in a general position.*

Proof. D' is the image of the locus D'' defined by the ordered six points $([l_1], \dots, [l_6]) \in (\mathbb{P}^2)^6$ such that $\widetilde{\Omega}([l_5], [l_6]) = 0$. Once we fix $([l_1], \dots, [l_5])$, the points $[l_6]$ are parameterized by the line $\widetilde{\Omega}([l_5], *) = 0$. Since $([l_1], \dots, [l_5])$ moves freely, the quintuples $([l_1], \dots, [l_5])$ are parameterized by $(\mathbb{P}^2)^5$. Then D'' is birational to a $\mathbb{P}^1$-bundle over $(\mathbb{P}^2)^5$. In particular, D'' is a prime divisor and so is D'.

Similarly, we can show that the sublocus in D'' consisting of sextuples $(l_1, \dots, l_6)$ not satisfying (1) or (2) is four-dimensional. Thus, the latter assertion follows. □

Lemma 6.8. *The restriction of $(D' \cap \pi^{-1}_{\mathrm{PGL}_3}(Y_2))/G$ to every fiber of $\pi^{-1}_{\mathrm{PGL}_3}(Y_2)/G \to Y_2$ is an open subset of $\mathbb{P}^4$.*

Proof. Let $l_1, \dots, l_6$ be six lines on B such that $([l_1], \dots, [l_6]) \in (\mathbb{P}^2)^6/\mathfrak{S}_6$ is mapped to a point y of Y_2. Let F be the fiber of $\pi^{-1}_{\mathrm{PGL}_3}(Y_2)/G \to Y_2$ over the point y. We show that the restriction of

$(D' \cap \pi_{\mathrm{PGL}_3}^{-1}(Y_2))/G$ to F is isomorphic to an open subset of $\mathbb{P}^4$. By Claim 2.1, G acts transitively on the set of general unordered pairs of intersecting lines. Therefore, a point of $F \cap D'$ is the image of a point $(g[l_1], \ldots, g[l_6]) \in (\mathbb{P}^2)^6/\mathfrak{S}_6$, where $g \in \mathrm{PGL}_3$ and $\widetilde{\Omega}(g[l_5], g[l_6]) = 0$. Now we choose a coordinate of $\mathbb{P}^2$ such that $\Omega = \{x^2 + y^2 + z^2 = 0\}$. Set $l_5 = (a_1 : a_2 : a_3)$ and $l_6 = (b_1 : b_2 : b_3)$. Then $\widetilde{\Omega}(g[l_5], g[l_6]) = 0$ if and only if

$$\begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} {}^t gg \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = 0. \tag{6.4}$$

Recall that by Claim 2.3 the map $\mathrm{PGL}_3 \to \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))) \simeq \mathbb{P}^5$ is defined by $g \mapsto {}^t gg$, where a conic on $\mathbb{P}^2$ is identified with a 3×3 symmetric matrix. Since condition (6.4) is linear on the entries of ${}^t gg$, $F \cap D'$ is an open subset of a hyperplane in $F \simeq \mathbb{P}^5$. □

We extend the involution J to some open set of $\pi_{\mathrm{PGL}_3}^{-1}(Y_2)/G$ including a suitable open set of $(D' \cap \pi_{\mathrm{PGL}_3}^{-1}(Y_2))/G$. For this purpose, we study the following locus $\mathcal{D}$ in $\mathcal{H}$ related to D'.

Condition 6.9. Let $\mathcal{D}$ be the locally closed subset of $\mathcal{H}$ consisting of sextic rational curves C that satisfy Conditions 3.20(a)–(f) and the following four conditions (g′)–(j′), which are modifications of Conditions 3.20(g)–(j):

(g′) Exactly two of six bi-secant lines, say β_1 and β_2, intersect. Note that $\beta_1 \cap \beta_2 \in C$ since two bi-secant lines of a normal rational curve intersect only on it (proof of Proposition 3.17). In this case, we also follow Notation 3.12. We can assume that $p_{11} = p_{21}$, which we denote by p. We also denote by ζ_1 the fiber of $E \to C$ over p.

(h′) For $i = 3, \ldots, 6$, there are two lines α_{i1} and α_{i2} intersecting both C and β_i outside $C \cap \beta_i$. For $i = 1, 2$, there is one line α_i intersecting both C and β_i outside $C \cap \beta_i$.

(i′) Any point in $\beta_i \cap C$ is not contained in B_φ. More explicitly, there are two lines different from β_i through each p_{ij} except p ($i = 1, \ldots, 6$, $j = 1, 2$), and there is one line different from β_1 and β_2 through p.
From this condition, it follows that no bi-secant lines of C are special lines.

(j′) For $i = 3, \ldots, 6$ and $j = 1, 2$, there are two lines γ_{ij1} and γ_{ij2} different from β_i such that they intersect both C and α_{ij} and their strict transforms on A intersect the strict transform of α_{ij}. For $i = 1, 2$, there are two lines γ_{i1} and γ_{i2} different from β_i such that they intersect both C and α_i and their strict transforms on A intersect the strict transform of α_i.

Note that by conditions (h′) and (i′), none of γ_{ij1}, γ_{ij2}, γ_{i1} and γ_{i2} intersects β_i.

Let

$$\widehat{\mathcal{H}}^\circ := \mathcal{H}^\circ \cup \mathcal{D},$$

which is an open subset of $\mathcal{H}^*$.

It is non-trivial to show that $\mathcal{D} \neq \emptyset$. For proof of this, the following degenerations of sextic normal rational curves are useful.

Lemma 6.10. *There exists a quintic normal rational curve C_5 and its uni-secant line l satisfying the following conditions:*

(1) *The hyperplane section H containing C_5 has one ordinary double point p as its singularity, and $p \in C_5$.*

(2) *Exactly two bi-secant lines l_1 and l_2 of C_5 pass through p.*
(3) *There exists another bi-secant line l_3 of C_5 and this is disjoint from l_1 and l_2.*
(4) *There exists another line passing through p and it is contained in H.*
(5) *l is not contained in H.*

Let $C_o := C_5 \cup l$. A general C_o satisfies the following additional conditions:

(A) *There exist three disjoint lines m_1, m_2, m_3 intersecting C_5 and l such that they are disjoint from l_1, l_2 and l_3 and $C_5 \cap l \neq C_5 \cap m_i$ $(i = 1, 2, 3)$.*
(B) *Conditions 6.9(a), (b), (f), (g′), (h′), (i′), and (j′) hold with obvious modifications.*

Proof. First we show that there is a quintic normal rational curve C_5 satisfying conditions (1)–(4). Let l_1, l_2, and l_3 be three general lines on B such that $l_1 \cap l_2 \neq \emptyset$. Considering the projection $B \dashrightarrow Q$ from l_3 (Proposition 2.6), we see that the strict transforms l_1' and l_2' are lines on Q such that $p' := l_1' \cap l_2' \neq \emptyset$. Take the tangent hyperplane section S of Q at p' (note that S is a quadric cone). Then the strict transform H on B of S has one ordinary double point p as its singularity, where p is the point corresponding to p'. Recall that the divisor T_{l_3} swept by lines intersecting l_3 is mapped by the projection to a twisted cubic γ on Q. The lines l_1' and l_2' intersect γ by Proposition 2.6. By the generality of l_1, l_2, and l_3, we can assume that S and γ intersect transversely at three points t, t_1 and t_2, where we can assume that $t_i = \gamma \cap l_i'$ $(i = 1, 2)$. Take a general twisted cubic curve C_5' in S passing though t. Note that $p' \in C_5'$. Then the strict transform C_5 on B of C_5' is a quintic normal rational curve satisfying conditions (1)–(4). Indeed, the lines l_1 and l_2 are bi-secant lines of C_5 since we can assume that $t_1, t_2 \notin C_5'$. The line through p as in condition (4) is the strict transform of the ruling of S passing through t. The bi-secant line l_3 of C_5 is the strict transform of the unique conic on S through t, t_1 and t_2. Take a general uni-secant line l of C_5. Then l is not contained in H. Hence, $C_5 \cup l$ satisfies conditions (1)–(5).

To obtain a general $C_5 \cup l$ satisfying (A) and (B), we only have to choose C_5 and l carefully. Checking is similar for each condition, so we only provide full details for (A) and sketch the checking for (B).

We only have to construct one C_o satisfying (A). Besides l_1, l_2 and l_3, we take three general lines m_1, m_2 and m_3 such that $[m_1], [m_2], [m_3]$ are collinear in $\mathbb{P}^2$. Then there is a line $L \subset \mathbb{P}^2$ containing $[m_1], [m_2]$ and $[m_3]$. By Proposition 2.5(3), there exists a line l on B such that $M(l) = L$. Therefore, $m_i \cap l \neq \emptyset$ $(i = 1, 2, 3)$. Consider the projection $B \dashrightarrow Q$ from l_3 as above. Let l', m_1', m_2' and m_3' be the strict transforms on Q of l, m_1, m_2 and m_3. Note that they are lines intersecting γ. By the generality of m_1, m_2 and m_3, we can assume that $H \cap l \neq H \cap m_i$ $(i = 1, 2, 3)$, and hence $S \cap l' \neq S \cap m_i'$ $(i = 1, 2, 3)$. Then we only have to take a twisted cubic C_5' on S passing through five points t, $S \cap l'$, $S \cap m_1'$, $S \cap m_2'$, and $S \cap m_3'$. It is quite easy to show the existence of such a twisted cubic by a simple dimension count on S. Note that since $S \cap l' \neq S \cap m_i'$ $(i = 1, 2, 3)$, we have $C_5' \cap l' \neq C_5' \cap m_i'$ and hence $C_5 \cap l \neq C_5 \cap m_i$ $(i = 1, 2, 3)$.

Now we sketch the verification of (B). We can verify conditions (h′), (i′) and (j′) using similar routines to those for the proof of (A). Condition (a) clearly holds for C_o since l is not contained in the hyperplane section containing C_5. Condition (b) is satisfied since C_o has exactly six bi-secant lines $l_1, \dots, l_3, m_1, \dots, m_3$. We check condition (f). For l_i $(i = 1, 2, 3)$, we can easily show that the normal bundles of its strict transforms on the blow-up of B along C_o are $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$ since the strict transform of the hyperplane section containing C_5 is smooth. For m_i $(i = 1, 2, 3)$, our previous argument [31, Lemma 5.1.6] works. Condition (g′) follows from conditions (2), (3) and (A). $\square$

Since C_o as in Lemma 6.10 satisfies Condition 6.9(b), $[C_o]$ is contained in the domain of the definition of Θ. Since C_o satisfies Condition 6.9(g′), the image of $[C_o]$ by Θ is contained in D'.

Lemma 6.11. *The morphism $\Theta_{|\widehat{\mathcal{H}}^\circ}$ is an isomorphism onto its image.*

Proof. The proof of Theorem 5.1 works with minor modifications in the last two paragraphs, so it is omitted. □

Proposition 6.12. *A general point of D' is the image by Θ of a point of $\mathcal{D}$. In particular, $\mathcal{D} \neq \emptyset$.*

Proof.

Step 1. We show that there exists a smoothing of a general C_o as in Lemma 6.10 that still has l_1 and l_2 as its bi-secant lines.

We use the notation as in the proof of Lemma 6.10. We set $C_5 \cap l_i = \{p, p_i\}$ $(i = 1, 2)$. Let $h: \widetilde{B} \to B$ be the blow-up of B at p, p_1 and p_2. Let $\widetilde{H}$, $\widetilde{C}$ and $\widetilde{l}$ be the strict transforms on $\widetilde{B}$ of H, C_5 and l, respectively. Then we see that $\widetilde{H}$ is smooth, $-K_{\widetilde{B}} \cdot \widetilde{C} = 4$ and $\widetilde{H} \cdot \widetilde{C} = 1$. Therefore, we see that $\mathcal{N}_{\widetilde{C}/\widetilde{B}} \simeq \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$. Since h is isomorphic near l, we have $\mathcal{N}_{\widetilde{l}/\widetilde{B}} \simeq \mathcal{N}_{l/B} \simeq \mathcal{O}^{\oplus 2}$. Then, by a previous argument [31, proof of Proposition 2.2.2], we see that $\widetilde{C} \cup \widetilde{l}$ is smoothable. The image on B of a smoothing is what we want.

Step 2. We show that a general smoothing C of C_o satisfies Conditions 6.9(a)–(f) and (g′)–(j′).

As for the conditions (a), (b), (f), (g′), (h′), (i′), and (j′), we only have to check them for a general C_o, which is already completed in Lemma 6.10(B). As usual, we denote by $\beta_1, \dots, \beta_6$ the six bi-secant lines of C and assume that $\beta_1 \cap \beta_2 \neq \emptyset$.

Now we verify the remaining conditions (c)–(e).

(c) By the construction of C_o as in the proof of Lemma 6.10, we can check that a general C_o intersects B_φ transversally at 12 points, and hence C does too. Therefore, $\varphi^{-1}(C)$ is a smooth curve of genus four, as in Proposition 3.11. Thus, by a previous argument [31, proof of Corollary 4.1.2], Sing $M(C)$ consists of six nodes that correspond to six bi-secant lines of C.

(d) We show that no three points among $[\beta_1], \dots, [\beta_6]$ are collinear. Assume the contrary. For a general C_o, only $[m_1]$, $[m_2]$ and $[m_3]$ among the sets of three points in $[l_1], \dots, [m_3]$ are collinear by the construction of C_o as in the proof of Lemma 6.10. Therefore, we can assume that $[\beta_4]$, $[\beta_5]$, $[\beta_6]$ are collinear. Since β_4, β_5 and β_6 are disjoint, the argument in the proof of Proposition 3.6 leads to a contradiction. We derive the argument that $[\beta_1], \dots, [\beta_6]$ are not on a conic since a similar property holds for the six bi-secant lines of $[C_o]$.

(e) By previous arguments [30, proofs of Lemma 3.1.1 and Proposition 3.1.2], it suffices to show that $[\beta_1], \dots, [\beta_6]$ are not on a conic, and $M(C)$ is not tangent to the conic Ω. The first condition follows from (d). We show the second condition. As in the verification of (c), a general C intersects B_φ transversally at 12 points. This implies that $M(C)$ and Ω intersect transversally at 12 points. This completes the proof.

Step 3. Finally, we show that $\mathcal{D} \to D'$ is dominant.

We only have to show that general $[C_o]$ curves form a family $\mathcal{C}$ such that $\Theta(\mathcal{C})$ is irreducible and is of codimension 2 in $(\mathbb{P}^2)^6/\mathfrak{S}_6$. Indeed, assume that we prove this assertion. Then general smoothing C of general C_o form a family of codimension 1 in $\mathcal{H}$. Therefore, by Lemma 6.11, $\Theta(\mathcal{D})$ is of codimension 1 in $(\mathbb{P}^2)^6/\mathfrak{S}_6$ and is contained in D'. Since D' is a prime divisor by Lemma 6.7, $\mathcal{D} \to D'$ is dominant.

Now we show that $\Theta(\mathcal{C})$ is irreducible and is of codimension 2 in $(\mathbb{P}^2)^6/\mathfrak{S}_6$. By the construction of C_o as in the proof of Lemma 6.10, the image of $[C_o]$ by Θ is the sextuple $([l_1], [l_2], [l_3], [m_1], [m_2], [m_3])$. Since l_1, l_2, l_3 are general under the condition $l_1 \cap l_2 \neq$

$\emptyset$, $([l_1],[l_2],[l_3])$ form a five-dimensional family. Moreover, since m_1, m_2, m_3 are general under the condition that $[m_1],[m_2],[m_3]$ are collinear, $([m_1],[m_2],[m_3])$ form a five-dimensional family. Therefore, $\Theta(G)$ is 10-dimensional, namely, is of codimension 2 in $(\mathbb{P}^2)^6/\mathfrak{S}_6$. Moreover, in a similar way to the proof of Lemma 6.7, we see that $\Theta(G)$ is irreducible. $\square$

Now we extend the involution J to the image of $\widehat{\mathcal{H}}^\circ$. For this purpose, we repeat the part from Lemma 4.8 up to Claim 4.15 with minor modifications. The structure of the part from Lemma 6.13 up to Claim 6.20 is parallel. Since the proofs of the assertions are almost the same with minor modifications, we only give precise statements and a few comments on the proofs.

Hereafter up to Claim 6.20, we assume that $[C] \in \mathcal{D}$. We basically follow the notation as in Lemma 4.8–Claim 4.15. Recall the notation in Condition 6.9. We denote by α'_i the strict transform on A of α_i $(i = 1, 2)$. Also note that the final assertion of Theorem 4.6 (Claim A.12) still holds for such a C.

Lemma 6.13. (1) *Let l be a line on B intersecting C. Assume that l is not a bi-secant line of C and that the strict transform l' of l on A intersects a flopping curve β'_i. Then $l = \alpha_{ij}$ $(j = 1, 2)$ or α_i as in Condition* 6.9 (h′)*, and α_{ij} and α_i do not intersect β_k $(k \neq i)$.*

(2) *The curves α'_{ij} $(i = 1, \dots, 6,\ j = 1, 2)$, α'_i $(i = 1, 2)$ and ζ_1 are fibers of f' intersecting flopped curves, and vice versa.*

(3) *$\widehat{\beta}_i$ is a bi-secant line of $\widehat{C}$. If $i \geq 3$, then it intersects $\widehat{C}$ transversely at the images of α'_{i1} and α'_{i2}. If $i = 1, 2$, then it intersects $\widehat{C}$ transversely at the images of α'_i and ζ_1.*

Proof. The assertion that ζ_1 is a fiber intersecting flopping curves follows in a similar way to the verification of (g) in the proof of Claim 4.10. $\square$

Claim 6.14. *For $i = 1, \dots, 6$, let $\widehat{\zeta}_{i1}$ and $\widehat{\zeta}_{i2}$ be the strict transforms on B on the right-hand side of* (4.2) *of ζ_{i1} and ζ_{i2} except ζ_1 (Notation* 3.12*). Then $\widehat{\zeta}_{i1}$ and $\widehat{\zeta}_{i2}$ are lines intersecting both $\widehat{C}$ and $\widehat{\beta}_i$ outside $\widehat{C} \cap \widehat{\beta}_i$. In particular, Condition* 6.9 (h′) *holds for $\widehat{C}$.*

Proof. The assertion follows in a similar way to the proof of Claim 4.9 by Condition 6.9(i′) for C. $\square$

Claim 6.15. *$\widehat{C}$ satisfies Conditions* 6.9(a), (b), (f), (g′), (i′) *and* (j′).

Lemma 6.16. *A line on A intersecting one β_i is one of the following, and similar statements hold for A':*

(1) *the strict transform α'_{ij} of α_{ij}* $(i \geq 3, 1 \leq j \leq 2)$,

(2) *the strict transform α'_i of α_i* $(i = 1, 2)$, *or*

(3) *the line l_{ij}* $(1 \leq j \leq 2)$ *as in Proposition* 3.14(2).

Lemma 6.17. *There exists a natural one-to-one correspondence between lines on A and lines on A' as follows:*

(1) *For a line on A disjoint from $\beta'_1, \dots, \beta'_6$, its strict transform on A' is a line on A' and vice versa.*

(2) *Fix one β_i. A line on A intersecting β'_i and a line on A' intersecting β''_i correspond to each other as follows:*

  (2-1) *If $i \geq 3$, then Lemma* 4.12 (2-1) *or* (2-2) *holds.*
  Assume $i = 1, 2$ below.

(2-2) *The line* α'_i *on* A *corresponds to the line* $\alpha'_i \cup \beta''_i$ *on* A'.

(2-3) *The line* l_{i2} *on* A *corresponds to* ζ_{i2} *on* A'.

(2-4) *The line* l_{i1} *on* A *corresponds to the line* $\zeta_1 \cup \beta''_i$ *on* A'.

Proof. Only case (2-4) is essentially new, but this follows from Lemmas 6.13 and 6.16. □

Claim 6.18. $\widehat{C}$ *satisfies Condition* 3.20(c). *Moreover,* $\widehat{\mathcal{H}}_1$ *is isomorphic to* $\mathcal{H}_1$*, where* $\widehat{\mathcal{H}}_1 := \varphi^{-1}(\widehat{C})$.

Lemma 6.19. *Let* g_i *be the unique conic on* $\mathbb{P}^2$ *passing through* $[\beta_1], \dots, [\check{\beta}_i], \dots, [\beta_6]$ *(Condition* 6.9(d)*). For* $3 \le i \le 6$, $[\alpha_{i1}], [\alpha_{i2}]$ *are precisely the intersection points of* $M(\beta_i)$ *and* g_i*. For* $i = 1, 2$, $[\alpha_i], [\beta_{3-i}]$ *are precisely the intersection points of* $M(\beta_i)$ *and* g_i.

Proof. For $3 \le i \le 6$, the proof of Lemma 4.14 still works. For $i = 1, 2$, we only have to replace α'_{i1} and α'_{i2} by α'_i and $l_{3-i\,1}$ in the proof of Lemma 4.14. □

Claim 6.20. $\widehat{C}$ *satisfies Conditions* 6.9(d) *and* (e).

Therefore, by Claims 6.14, 6.15 *and* 6.18*, we have* $[\widehat{C}] \in \mathcal{D}$.

Proof. The proof is the same as that for Claim 4.15, except that to check (d) we need modifications according to Lemmas 6.17 and 6.19. □

By Claim 6.20, the involution on $\widetilde{\mathcal{S}}_4^{+\circ}$ extends naturally to the image of $\widehat{\mathcal{H}}^\circ$ in $\widetilde{\mathcal{S}}_4^{+}$.

We denote by

$$\widehat{V}_1 \subset U/\mathfrak{S}_6$$

the image of $\widehat{\mathcal{H}}^\circ$ by Θ. By Lemma 6.11, we can extend the involution J to $\widehat{V}_1$. We denote the extension by J also. Note that the image of $\mathcal{D}$ is preserved by J. Let

$$\widehat{W} := (\pi_{\mathrm{PGL}_3}^{-1}(Y_2)/G) \cap (\widehat{V}_1/G),$$

which is preserved by J. We denote the restriction of J to $\widehat{W}$ by J also.

Denote by

$$D \subset \widehat{W}$$

the restriction of $(D' \cap \pi_{\mathrm{PGL}_3}^{-1}(Y_2))/G$. In other words,

$$\widehat{W} = W \cup D.$$

Note that $\overline{W}$ as in Proposition 6.6 contains $\widehat{W}$ as an open subset since it contains $\pi_{\mathrm{PGL}_3}^{-1}(Y_2)/G$.

Lemma 6.21. *The involution* J *on* $\widehat{W}$ *extends to* $\overline{W}$.

Proof. Let $\overline{D}$ be the closure of D in $\overline{W}$. Then the restriction to $\overline{D}$ of $\overline{\varrho}'$ is a sub-$\mathbb{P}^4$-bundle. By [13, III Corollary 12.9], $\mathcal{F} = \overline{\varrho}'_* \mathcal{O}_{\overline{W}}(\overline{D})$ is a locally free sheaf of rank 6 on Y_2 and

$\overline{\varrho}'_* \mathcal{O}_{\overline{W}}(\overline{D}) \otimes k(y) \simeq H^0(\overline{\varrho}'^{-1}(y), \mathcal{O}_{\overline{W}}(\overline{D})_{|\varrho'^{-1}(y)})$ for $y \in Y_2$. Consider the following diagram:

$$\begin{tikzcd}
\overline{\varrho}'^* \mathcal{F} \arrow[rr] \arrow[d] & & \mathcal{O}_{\overline{W}}(\overline{D}) \arrow[d] \\
\overline{\varrho}'^* \mathcal{F}_{|\overline{\varrho}'^{-1}(y)} \arrow[rr] \arrow[d] & & \mathcal{O}_{\overline{W}}(\overline{D})_{|\overline{\varrho}'^{-1}(y)} \arrow[d, "id"'] \\
H^0(\overline{\varrho}'^{-1}(y), \mathcal{O}_{\overline{W}}(\overline{D})_{|\overline{\varrho}'^{-1}(y)}) \otimes \mathcal{O}_{\overline{\varrho}'^{-1}(y)} \arrow[rr] & & \mathcal{O}_{\overline{W}}(\overline{D})_{|\overline{\varrho}'^{-1}(y)}.
\end{tikzcd}$$

The map $H^0(\overline{\varrho}'^{-1}(y), \mathcal{O}_{\overline{W}}(\overline{D})_{|\overline{\varrho}'^{-1}(y)}) \otimes \mathcal{O}_{\overline{\varrho}'^{-1}(y)} \to \mathcal{O}_{\overline{W}}(\overline{D})_{|\overline{\varrho}'^{-1}(y)}$ is surjective since $\overline{\varrho}'^{-1}(y) \simeq \mathbb{P}^5$ and $\mathcal{O}_{\overline{W}}(\overline{D})_{|\overline{\varrho}'^{-1}(y)} \simeq \mathcal{O}_{\mathbb{P}^5}(1)$. Thus, by the Nakayama lemma, $\overline{\varrho}'^* \mathcal{F} \to \mathcal{O}_{\overline{W}}(\overline{D})$ is surjective. By [13, II Proposition 7.12], a morphism $\gamma : \overline{W} \to \mathbb{P}(\mathcal{F})$ over Y_2 remains defined. Since γ is fiberwise an isomorphism, then it is an isomorphism by the main Zariski theorem.

Let Y° be any open subset of Y_2. Since D is invariant under the involution J, it holds that $\Gamma(Y^\circ, \mathcal{F}) \simeq \Gamma(j(Y^\circ), \mathcal{F})$, which induces an isomorphism $\mathcal{F} \simeq j^*\mathcal{F}$. Thus, J extends to the involution $\overline{W} \simeq \mathbb{P}(j^*\mathcal{F}) \to \mathbb{P}(\mathcal{F}) = \overline{W}$. □

6.5. End of the proof

We still denote by J the extension of J to $\overline{W}$. Now we can prove the main result.

Theorem 6.22. *$\overline{W}/J$ is a rational variety.*

Proof. The action of J is trivial on the fiber of $\overline{\varrho}'$ since J is an involution and j acts non-trivially on W_2. Thus, $\overline{\varrho}'$ descends to a $\mathbb{P}^5$-bundle $p : \overline{W}/J \to Y_2/j$. Moreover, the sub-$\mathbb{P}^4$-bundle $\overline{D}$ of $\overline{W}$ descends to a sub-$\mathbb{P}^4$-bundle G of $\overline{W}/J$ since it is invariant under J. Set $\mathcal{E} := p_* \mathcal{O}_{\overline{W}/J}(G)$. In the similar way to the proof of Lemma 6.21, we can show that $\overline{W}/J \simeq \mathbb{P}(\mathcal{E})$. In particular, $\overline{W}/J$ is a locally trivial $\mathbb{P}^5$-bundle over W_2/j. Consequently, $\overline{W}/J$ is rational since Y_2/j is rational by Theorem 6.2. □

Corollary 6.23. *$\mathcal{S}_4^+$ is a rational variety.*

Proof. The proof follows from Proposition 5.2 and Theorem 6.22. □

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Appendix. Proof of Theorem 4.6

In the proof of Theorem 4.6, we use the following special birational map originating from B.

Definition A.1. Let b be a point of B. We call the rational map from B defined by the linear system of hyperplane sections singular at b *the double projection from b*.

Proposition A.2. *For a point $b \in B - B_\varphi$, the double projection from b is described as follows:*

(1) *The target of the double projection is $\mathbb{P}^2$, and the double projection from b and the projection $B \dashrightarrow \overline{B}_b$ from b fit into the following diagram:*

$$\begin{tikzcd}
 & B_b \arrow[dl, "\pi_{1b}"'] \arrow[dr] \arrow[rr, dashed] & & B'_b \arrow[dl] \arrow[dr, "\pi_{2b}"] & \\
B & & \overline{B}_b & & \mathbb{P}^2,
\end{tikzcd}$$

where π_{1b} is the blow-up of B at b, $B_b \dashrightarrow B'_b$ is the flop of the strict transforms of three lines through b, and $\pi_{2b}\colon B'_b \to \mathbb{P}^2$ is a (unique) $\mathbb{P}^1$-bundle structure.
We denote by E_b the π_{1b}-exceptional divisor. For simplicity, we denote the strict transforms on B'_b of divisors on B_b using the same notation.

(2)

$$L_b = H_b - 2E_b \quad \text{and} \quad -K_{B'_b} = H_b + L_b,$$

where H_b is the strict transform of a general hyperplane section of B, and L_b is the pull-back of a line on $\mathbb{P}^2$.

(3) *The strict transforms l'_i of three lines l_i through b on B_b have the normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. The flop $B_b \dashrightarrow B'_b$ is Atiyah's flop.*

(4) *A fiber of π_{2b} not contained in E'_b is the strict transform of a conic through b, the strict transform of a line $\not\ni b$ intersecting a line through b.*

Proof. See [11], for example. $\square$

Now we begin our proof of Theorem 4.6. The proof consists of several claims.

We frequently use the following basic numerical equalities:

$$H^3 = 5, \qquad H^2E = 0, \qquad HE^2 = -6 \quad \text{and} \quad E^3 = -10. \tag{A.1}$$

In applying the so-called two-ray game, the following claim is basic.

Claim A.3. *A is a weak Fano threefold, namely, $-K_A$ is nef and big.*

Proof. For the nefness of $-K_A$, we can actually prove more; $|-K_A| = |2H - E|$ is base point-free since C is the intersection of quadrics. The bigness of $-K_A$ follows from the calculation

$$(-K_A)^3 = (f^*(-K_B) - E)^3 = (2H - E)^3 = 8H^3 + 6HE^2 - E^3 = 14 > 0. \quad \square$$

Let $g\colon A \to \overline{A}$ be the Stein factorization of the morphism defined by $|-K_A|$.

Claim A.4. *g does not contract a divisor.*

Proof. Assume that g contracts a divisor F, which is prime since $\rho(A/\overline{A}) = 1$. We can write $F \sim aH - bE$, where $a, b \in \mathbb{Z}$. It holds that $(-K_A)^2 F = 0$. By $-K_A = 2H - E$ and (A.1), we have $(-K_A)^2 F = 14(a - b) = 0$. Thus, $F = a(H - E)$. The image $g(F)$ of F is not a point since $-K_A F^2 = -4a^2 \neq 0$. For a fiber l of $F \to g(F)$, it holds that $F \cdot l = -1$ or -2 by adjunction. If $F \cdot l = -1$, then $a = 1$ and $F \sim H - E$. This is impossible; $|H - E|$ is empty since C is not contained in a hyperplane section. Thus, $F \cdot l = -2$ and $F \sim 2(H - E)$. Taken together with the equality $-K_A \cdot l = (2H - E) \cdot l = 0$, it holds that $H \cdot l = 1$ and $E \cdot l = 2$, namely, l is irreducible and is the strict transform of a bi-secant line of C. This is impossible since C has only a finite number of bi-secant lines. □

Therefore, g is a flopping contraction. Moreover, it holds that $\rho(A/\overline{A}) = 1$ since $\rho(A) = 2$. Let $A \dashrightarrow A'$ be the flop. Since A' is rational, $K_{A'}$ is not nef. This implies that there exists an extremal contraction $f' : A' \to B'$. The morphism f' is unique since $\rho(A') = 2$. For simplicity, we denote the strict transforms on A' of curves and divisors on A using the same notation. We show that f' is defined by the linear system associated with some sufficiently high multiple of

$$L := 3H - 2E.$$

We need to show some auxiliary claims.

Claim A.5. *Let D be an effective divisor on A. We write $D \sim aH - bE$ with some $a, b \in \mathbb{Z}$. Then $a = 0$ and $b \leq 0$, or $a > 0$ and $b < a$.*

Proof. We have $a \geq 0$ since H is big and E is not big. If $a = 0$, then clearly we have $b \leq 0$.

Assume by contradiction that $a > 0$ and $b \geq a$. Then $(-K_A)^2 D \leq 0$ by (A.1), and hence $(-K_A)^2 D = 0$ since $-K_A$ is nef, and then D is the g-exceptional divisor. This contradicts Claim A.4. □

Claim A.6. $h^0(\mathcal{O}_A(L)) \geq 7$.

Proof. Consider the exact sequence

$$0 \to \mathcal{O}_A(L) \to \mathcal{O}_A(3H - E) \to \mathcal{O}_E(3H - E) \to 0. \tag{A.2}$$

$3H - E$ is nef since $3H - E = 2H - E + H = -K_A + H$, and $-K_A$ and H are nef. Thus, by the Kawamata–Viewheg vanishing theorem, $h^0(\mathcal{O}_A(3H - E)) = \chi(\mathcal{O}_A(3H - E)) = \frac{1}{12}(120H^3 + 49HE^2 - 6E^3) + \frac{1}{12}H \cdot c_2(A) + 3$. Let $H_0 \in |H|$ be a general member. By the exact sequence $0 \to T_{H_0} \to T_{A|H_0} \to \mathcal{O}_{H_0}(H) \to 0$, we can calculate $c_2(A) \cdot H = 18$. Thus, by (A.1), we have $h^0(\mathcal{O}_A(3H - E)) = 35$. Now we compute $h^0(\mathcal{O}_E(3H - E))$. Note that E is a $\mathbb{P}^1$-bundle over $C \simeq \mathbb{P}^1$. Let l be a fiber of $E \to C$. Then $(3H - E) \cdot l = 1$. Thus, $f_{|E*}\mathcal{O}_E(3H - E) = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$, where $a + b = (3H - E)^2 E = 26$ and $a, b \geq 0$ since $3H - E$ is nef. Therefore, $h^0(\mathcal{O}_E(3H - E)) = 28$. Finally, we have $h^0(\mathcal{O}_A(L)) \geq 7$ from (A.2). □

Claim A.7. *L is nef on A'.*

Proof. We show that $|L|$ has no fixed component. Assume by contradiction that $|L|$ has a fixed component. If E is a fixed component, then $L - E \sim 3H - 3E$ is effective, a contradiction to Claim A.5. If there exists a fixed component $D \sim aH - bH$ with $a > 0$ and $b < a$, then

$L - (aH - bE) = (3 - a)H - (2 - b)E$ is effective. By Claim A.6, $h^0(L - D) \geq 7$, and thus $3 - a > 0$ and $2 - b < 3 - a$ by Claim A.5. The inequality $b < a$ and $2 - b < 3 - a$ has no solution, which is a contradiction. Therefore, $|L|$ has no fixed component.

Now we prove that L is nef on A'. Since $\rho(A') = 2$, it suffices to check that L is non-negative for both a flopped curve and a general curve in a general fiber of f'. First, we check that L is positive for a flopped curve on A'. Indeed, for a flopping curve γ, it holds that $H \cdot \gamma > 0$ and $(2H - E) \cdot \gamma = -K_A \cdot \gamma = 0$. Thus, $L \cdot \gamma = (3H - 2E) \cdot \gamma < 0$ on A. Then, by Proposition 4.2, L is positive for a flopped curve on A'. Second, we check that L is non-negative for a general curve in a general fiber of f'. If f' is of fiber type, then curves in fibers cover A', whence L is non-negative for a general curve in a general fiber of f' since $|L| \neq \emptyset$ by Claim A.6. If f' is birational, then, again, L is non-negative for a general curve in a general fiber of f' since the f'-exceptional divisor is not a fixed component of $|L|$ on A'. □

Now we can show that $|mL|$ ($m \gg 0$) defines f' by the following claim.

Claim A.8. *L is not ample.*

Proof. We only have to find a curve numerically trivial for L. We see that an irreducible tri-secant conic of C suffices for this purpose. We show its existence by the double projection from a general point b of C (Proposition A.2).

By the assumption, C is not contained in B_φ. Therefore, $b \notin B_\varphi$ and then there are three lines l_1, l_2 and l_3 through b. We consider the double projection from b and we use the notation of Proposition A.2. Since C has only finitely many bi-secant lines, we can assume that l_i are not bi-secant lines by the generality of b. Thus, the strict transforms C' and l'_i of C and l_i are disjoint on B_b. By $-K_{B_b} = \pi_{1b}^*(-K_B) - 2E_b$, it holds that $-K_{B_b} \cdot C' = 10$. Therefore, it holds that $H_b \cdot C' = 6$ on B'_b and $-K_{B'_b} \cdot C' = 10$, where we denote by C' the strict transform on B'_b of C', by abuse of the notation. Hence, $L_b \cdot C' = 4$ by Proposition A.2(2) and then the image of C' on $\mathbb{P}^2$ is a line, a conic or a quartic. This implies that π_{2b} has a multi-secant fiber q' of C'. Indeed, if the image of C' on $\mathbb{P}^2$ is a line or a conic, then $\pi_{2b|C}$ is not birational, and thus any fiber of π_{2b} intersecting C' is a multi-secant fiber of C'. If the image of C' on $\mathbb{P}^2$ is a quartic C'', then C'' is singular since C'' is rational, and thus the fiber of π_{2b} over a singular point of C'' is a multi-secant fiber of C'.

The possibilities for q' are as in statement (4) of Proposition A.2. We see that q' is not contained in E'_b since C' intersects E'_b at one point. If this fiber is the strict transform of a smooth conic q through b, then q is a k-secant conic of C with $k \geq 3$. Otherwise, the fiber is the strict transform of a bi-secant line of C intersecting one of l_i. We show that this does not occur if b is general. If this occurs for general b, then C is contained in the locus of lines T_β intersecting one fixed bi-secant line β since there are a finite number of bi-secant lines of C. By Proposition 2.5(4), T_β is a hyperplane section of B. This is a contradiction since C is not contained in a hyperplane section. Therefore, there exists an irreducible k-secant conic of C with $k \geq 3$.

Let q be a general irreducible k-secant conic of C with $k \geq 3$. Then $L \cdot q = 6 - 2k$ on A. Since a flopping curve of $A \dashrightarrow A'$ intersects L negatively, we have $L \cdot q \leq 6 - 2k$ on A' by Proposition 4.4(2). Since L is nef on A' by Claim A.7, we have $k = 3$ and $L \cdot q = 0$ on A'. Thus, L is not ample. □

By the Kawamata–Shokurov base point-free theorem [16, Theorem 3-1-1], some sufficiently high multiple of L defines a morphism, which is non-trivial since L is not ample. The extremal contraction f' is nothing but this morphism.

Now we determine the type of f'. Note that L is the pull-back of a generator of Pic B' since L is primitive.

Claim A.9. *f' is not of fiber type.*

Proof. Suppose by contradiction that f' is of fiber type. Then $B' \simeq \mathbb{P}^1$ or $\mathbb{P}^2$. We can derive this fact as follows: it is well known that B' is smooth if f' is of fiber type [20]. Since A is rational, B' is covered by rational curves, and thus is rational since $\dim B' \leq 2$. If $\dim B' = 1$, then $B' \simeq \mathbb{P}^1$. If $\dim B' = 2$, then $B' \simeq \mathbb{P}^2$ since the Picard number of B' is 1. Thus, L is the pull-back of a point or a line, respectively. This is a contradiction since $h^0(L) \geq 7$ by Claim A.6. □

Claim A.10. *f' contracts a divisor E' to a smooth curve $\widehat{C}$. B' is the smooth quintic del Pezzo threefold.*

Proof. Let E' be the f'-exceptional divisor. Since f'_*L is the ample generator of Pic B', we can write $f'^*(-K_{B'}) = pL$, where p is the Fano index of B'. We write $-K_{A'} = f'^*(-K_{B'}) - dE'$, where d is the discrepancy. Then we have $2H - E = p(3H - 2E) - dE'$. Since E' is effective and is different from E, we have $3p - 2 > 2p - 1$ by Claim A.5. Thus, $p > 1$. By the classification of $\mathbb{Q}$-Fano threefolds with Fano index >1 [10,27] and $h^0(f'_*L) \geq 7$ (Claim A.6), B' must be a (possibly singular) quintic del Pezzo threefold. Then, by the classification of divisorial contractions from smooth projective threefolds [20], f' is one of the following:

E_1: f' is the blow-up of B' along a smooth curve $\widehat{C}$, or

E_2–E_4: f' is the blow-up at a point b of B'. More precisely,

E_2: b is a smooth point of B'. $E' \simeq \mathbb{P}^2$ and $-K_{A'|E'} = \mathcal{O}_{\mathbb{P}^2}(2)$,

E_3: B' is analytically isomorphic to $\{xy + zw = 0\} \subset \mathbb{C}^4$ near b. $E' \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and $-K_{A'} = f'^*(-K_{B'}) - E'$, or

E_4: B' is analytically isomorphic to $\{xy + z^2 + w^3 = 0\} \subset \mathbb{C}^4$ near b. E' is a singular quadric surface and $-K_{A'} = f'^*(-K_{B'}) - E'$.

Recall that $(-K_{A'})^3 = (-K_A)^3 = 14$. If f' is of type E_2, then $(-K_{A'})^3 = (-K_{B'})^3 - 8 = 40 - 8 = 32$, a contradiction. If f' is of type E_3 or E_4, then $(-K_{A'})^3 = (-K_{B'})^3 - 2 = 40 - 2 = 38$, a contradiction.

Therefore, f' is of type E_1 and B' is the smooth quintic del Pezzo threefold. □

To check the equalities in (4.3) is easy. By the definition of L, we have the first two equalities. By $-K_{A'} = 2L - E'$, $-K_{A'} = 2H - E$ and $L = 3H - 2E$, we have the third equality.

Claim A.11. *$\widehat{C}$ is a sextic normal rational curve.*

Proof. The following is a standard result for the blow-up of a smooth threefold along a smooth curve:

$$(-K_{A'})(E')^2 = 2g(\widehat{C}) - 2, \qquad (-K_{A'})^2 E' = (-K_{B'} \cdot \widehat{C}) + 2 - 2g(\widehat{C}).$$

By Proposition 4.4 and (4.3), we have $(-K_{A'})(E')^2 = (-K_A)(4H - 3E)^2$ and $(-K_{A'})^2 E' = (-K_A)^2(4H - 3E)$. By the equalities $-K_A = 2H - E$ and (A.1), we can easily show that $(-K_A)(4H - 3E)^2 = -2$ and $(-K_A)^2(4H - 3E) = 14$. Thus, $\widehat{C}$ is a smooth sextic rational curve. We show that $\widehat{C}$ is not contained in a hyperplane section. Assume by contradiction that $\widehat{C}$ is contained in a hyperplane section M. Then $f'^*M - E' \sim (3H - 2E) - (4H - 3E) = -H + E$ is effective, a contradiction. □

Claim A.12. *If* $[C] \in \mathcal{H}^\circ$*, then* C *has only a finite number of bi-secant lines and* C *is not contained in* B_φ*. Moreover, the final assertion of Theorem 4.6 holds.*

Proof. C has only a finite number of bi-secant lines by Condition 3.20(b). Assume by contradiction that $C \subset B_\varphi$. Then $\Omega \subset M(C)$ since B_φ is covered by special lines. In particular, $M(C)$ is not irreducible, a contradiction to Condition 3.20(c).

We show the last assertion of Theorem 4.6. Note that any β'_i is a g-exceptional curve. We show that β'_i $(1 \le i \le 6)$ are the only g-exceptional curves. Passing to the analytical category and taking the algebraization, we can decompose the flop $A \dashrightarrow A'$ into a sequence of flops $A := A_1 \dashrightarrow A_2 \dashrightarrow \cdots \dashrightarrow A_n =: A'$ for some $n \in \mathbb{N}$, where $A_j \dashrightarrow A_{j+1}$ is the flop of the strict transform of β'_j if $1 \le j \le 6$, or the flop of the strict transform of an irreducible g-exceptional curve different from β'_i $(1 \le i \le 6)$ if $6 < j \le n-1$. For simplicity, we denote the strict transforms of g-exceptional curves, and divisors L and H on each A_j using the same notation. Noting that $L = 3H - 2E$, we can easily compute that $L^3 = -1$ on A. Since L on A' is the pull-back of $\widehat{L}$, we have $L^3 = 5$ on A'. Note that the flop $A_j \dashrightarrow A_{j+1}$ $(1 \le j \le 6)$ is Atiyah's flop. Thus, by the equality $L \cdot \beta'_i = -1$ $(1 \le i \le 6)$ on A, we see that $L^3 = -1 + 6 = 5$ on A_7 by Proposition 4.5. Assume by contradiction that there exists at least one g-exceptional curve different from β'_i's, namely, $n > 7$. Since the strict transforms of all the other g-exceptional curves are still numerically negative for L on A_j $(j \ge 7)$ by Proposition 4.4(2), it holds that $L^3 > 5$ on $A' = A_n$ by Proposition 4.4(2), again a contradiction. Thus, β'_i $(1 \le i \le 6)$ are the only g-exceptional curves. $\square$

This ends the proof of Theorem 4.6.

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